Maximum likelihood SAR tomography based on the polarimetric multi-baseline RVoG model:

Optimal estimation of a covariance matrix structured as the sum of two Kronecker products.

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Outline

PolTomSAR using the MB-RVOG model and Kronecker products

Maximum Likelihood estimation of SKP-2 covariance matrices

Performance of the proposed ML estimator
Single-pol MB-RVOG model

SB case

- Uncorrelated responses
  \[ y_i(l) = s_{gi}(l) + s_{vi}(l), \quad l_i = l_g + l_v \]

- Homogeneous media
  \[ E(s_{x_1}s_{x_2}^*) = l_x\gamma_x, \quad \gamma_x = \int f_x(z)\exp(jk_z z)dz \]

- Global InSAR response
  \[ y(l) = [s_1(l), s_2(l)]^T \]
  \[ R_{SB} = l_gR_g + l_vR_v, \quad \text{with} \quad R_x = \begin{bmatrix} 1 & \gamma_x^* \\ \gamma_x & 1 \end{bmatrix} \]
Single-pol MB-RVOG model

MB case

- Point-like scatterer MB-response: \( s(l) = A_c(l)a(z) \)
  \[ a = [1, \exp(jkz_2z), \ldots, \exp(jkz_{Ns}z)] \]

- Global MB-InSAR response \( y(l) = s_g(l) + s_v(l) \)
  \[ R_{MB} = l_gR_g + l_vR_v \text{ with } R_x = \begin{bmatrix} 1 & \gamma_x(kz_2) & \cdots & \gamma_x(kz_{Ns}) \\ \gamma_x(kz_2) & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_x(kz_{Ns}) & \cdots & \cdots & 1 \end{bmatrix} \]
Single-pol MB-RVOG model

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\[
R_{MB} = l_gR_g + l_vR_v \quad \text{with} \quad R_x = \\
\begin{bmatrix}
1 & \gamma_x(k_{z_2}) & \ldots & \gamma_x(k_{z_{Ns}}) \\
\end{bmatrix}
\]

😊 Estimated profiles match LiDAR.

LiDAR — TomSAR
Full-pol MB-RVOG model

- Point-like scatterer MB-response: \( y(l) = A_c(l) k_h a(z) \)
Full-pol MB-RVOG model

- Point-like scatterer MB-response: \[ y(l) = A_c(l) \begin{bmatrix} k_{hh} a(z) \\ k_{hv} a(z) \\ k_{vv} a(z) \end{bmatrix} = A_c(l) k \otimes a(z) \]

- Uncorrelated P-S processes: \[ E(y_i y_i^\dagger) = I_i C_i \otimes R_i \]
Full-pol MB-RVOG model

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- Global MB-POLinSAR response: \( y(l) = y_g(l) + y_v(l) \)
  \[ R_{PS} = C_g \otimes R_g + C_v \otimes R_v \]
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\[ R_{PS} = C_g \otimes R_g + C_v \otimes R_v \]

- SB-case (\( N_s = 2 \))

\[ R_{SP} = R_g \otimes C_g + R_v \otimes C_v = \begin{bmatrix} C_g + C_v & \gamma_g C_g + \gamma_v C_v \\ \gamma_g^* C_g^\dagger + \gamma_v^* C_v^\dagger & C_g + C_v \end{bmatrix} \]

\[ R_{SP} = \begin{bmatrix} C & C_{12} \\ C_{12}^\dagger & C \end{bmatrix} \]

\[ \gamma(w) = \frac{w^\dagger C_{12} w}{w^\dagger C w} = \frac{\gamma_v + \mu(w) \gamma_g}{1 + \mu(w)} \]

MB-RVOG: exactly modeled by an SKP-2 structured covariance matrix

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MB-RVOG parameter estimation

**SB case**
- Coherence line estimation
- GV separation: strong assumptions
  - $|\gamma_g| = 1$, rank$(C_g) < 3$
  - $f_v(z)$ known

**MB case**
- SVD SKP-2 decomposition
- GV separation:
  - several solutions
  - accuracy increases with $N_s$
MB-RVOG parameter estimation

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See also: physical interpretation of MB-RVOG JD & SKP decomp.
B. El Hajj-Chehade, L. Ferro-Famil, Poster session Wednesday aft.
Estimation problem statement and objectives

- L independent observations \( \{y(l)\}_{l=1}^{L} \)
- Estimate RVOG parameters or its SKP-2 covariance matrix

**SB case**

- Pol-Sampling \( \gamma(w_i) \), line LS estimate [C & P 2001-3][Flynn & Tabb 2002-3]
- LS estimation (matrix formatting) [LFF et al, 2009-10] [L-M 2012]
- ML estimation [Flynn et al. 2002] and CRLB [Roueff et al. 2011].

**MB case**

- LS estimation using joint diagonalization [Ferro-Famil et al. 2010]
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**LS vs ML estimation**

- Estimate that best fits the observations (LS) vs Estimate most likely to have produced observations (ML).
- LS : generally fast, often has analytical solutions, no pdf assumption
- ML : if UMV estimator exists → ML. Can be complex and time consuming.
ML estimation of SKP2 covariance matrices

Objectives

- Find $\hat{R}_{PS-ML} = \arg \max_{\hat{R}_{PS}} f(\{y(l)\}_{l=1}^{L} | R_{PS})$ under the SKP-2 constraint.

- ML (Wishart) concentrated criterion:

$$J(R_{PS}) = \text{tr}(R_{PS}^{-1} \hat{R}_{yy}) + \log |R_{PS}| \quad \text{with} \quad \hat{R}_{yy} = \frac{1}{L} \sum_{l=1}^{L} y_{PS}(l)y_{PS}^{\dagger}(l)$$
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- Estimation objective : determine 
  
  $$\hat{R}_{PS-ML} = \arg \min J(R_{PS}) \text{ with and } R_{PS} = C_g \otimes R_g + C_v \otimes R_v$$

- Find a fast and reliable technique
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- Find a fast and reliable technique

Case of a single KP

- No analytical solution.
- Fast and reliable iterative alternate (Flip-Flop) optimization [Lu 2004]
- Asymptotic analytical expression : Weighted LS solution [Werner 2008]
ML estimation of SKP2 covariance matrices
SKP-2 case

- In general $A \otimes B + C \otimes D \neq E \otimes F$
- No existing solution.
ML estimation of SKP2 covariance matrices

SKP-2 case

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- No existing solution.

Exact joint diagonalization of two matrices

- $A, B > 0$ and Hermitian can be exactly jointly diagonalized [Yeredor 2005] :
  \[ \exists W : WAW^\dagger = D_A, \ WBW^\dagger = D_B, \text{ in general } WW^\dagger \neq I \]
ML estimation of SKP2 covariance matrices

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- In the SKP-2 case, $R_{PS} = C_g \otimes R_g + C_v \otimes R_v$
  $$\exists W = W_C \otimes W_R : \; \quad WR_{PS}W^\dagger = D_{C_g} \otimes D_{R_g} + D_{C_v} \otimes D_{R_v} = D$$
ML estimation of SKP2 covariance matrices

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ML criterion minimization over $D$

- Inserting $D, W$ into $J(R_{PS})$
  \[ J(D, W) = \text{tr}(D^{-1}W\hat{R}_{yy}W^\dagger) + \log|D| - \log|WW^\dagger| \]
- Minimum obtained for : $\hat{D} = \text{diag}(W\hat{R}_{yy}W^\dagger)$
ML estimation of SKP2 covariance matrices

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ML criterion minimization over \( D \)

- Inserting \( D, W \) into \( J(R_{PS}) \)
  \[ J(D, W) = \text{tr}(D^{-1}WR_{yy}W^\dagger) + \log |D| - \log |WW^\dagger| \]
- Minimum obtained for: \( \hat{D} = \text{diag}(WR_{yy}W^\dagger) \)
- Resulting concentrated ML criterion
  \[ J(W) = \log |\text{diag}(WR_{yy}W^\dagger)| - \log |WW^\dagger| \]
ML estimation of SKP2 covariance matrices

ML criterion minimization over $\mathbf{W}$: formulation

- Explicit ML criterion $J(\mathbf{W}_C, \mathbf{W}_R) = \log |\text{diag}(\tilde{D})| - \log |\mathbf{W}\mathbf{W}^\dagger|$

  with $\tilde{D} = \mathbf{W}\hat{\mathbf{R}}_{yy}\mathbf{W}^\dagger$ and $\mathbf{W} = \mathbf{W}_C \otimes \mathbf{W}_R$
ML estimation of SKP2 covariance matrices

ML criterion minimization over $\mathbf{W}$ : formulation

- Explicit ML criterion $J(\mathbf{W}_C, \mathbf{W}_R) = \log |\text{diag}(\tilde{D})| - \log |\mathbf{W}\mathbf{W}^\dagger|$
  
  with $\tilde{D} = \mathbf{W}\hat{\mathbf{R}}_{yy}\mathbf{W}^\dagger$ and $\mathbf{W} = \mathbf{W}_C \otimes \mathbf{W}_R$

- Conditional criterion : known spatial diagonalizer
  
  $J(\mathbf{W}_C|\mathbf{W}_R) = \log |\sum_{i=1}^{N_s} \text{diag}(\mathbf{W}_C\tilde{D}_C^i\mathbf{W}_C)| - \log |\mathbf{W}_C\mathbf{W}_C^\dagger|$

- Conditional criterion : known polarimetric diagonalizer
  
  $J(\mathbf{W}_R|\mathbf{W}_C) = \log |\sum_{i=1}^{N_p} \text{diag}(\mathbf{W}_R\tilde{D}_R^i\mathbf{W}_R)| - \log |\mathbf{W}_R\mathbf{W}_R^\dagger|$
ML estimation of SKP2 covariance matrices

ML criterion minimization over $\mathbf{W}$ : formulation

- Explicit ML criterion $J(\mathbf{W}_C, \mathbf{W}_R) = \log |\text{diag}(\tilde{D})| - \log |\mathbf{W}\mathbf{W}^\dagger|$ with $\tilde{D} = \mathbf{W}^\dagger\mathbf{R}_{yy}\mathbf{W}^\dagger$ and $\mathbf{W} = \mathbf{W}_C \otimes \mathbf{W}_R$

- Conditional criterion : known spatial diagonalizer
  $$J(\mathbf{W}_C|\mathbf{W}_R) = \log |\sum_{i=1}^{N_s} \text{diag}(\mathbf{W}_C\tilde{D}_c, \mathbf{W}_C)| - \log |\mathbf{W}_C\mathbf{W}_C^\dagger|$$

- Conditional criterion : known polarimetric diagonalizer
  $$J(\mathbf{W}_R|\mathbf{W}_C) = \log |\sum_{i=1}^{N_p} \text{diag}(\mathbf{W}_R\tilde{D}_r, \mathbf{W}_R)| - \log |\mathbf{W}_R\mathbf{W}_R^\dagger|$$

ML criterion minimization over $\mathbf{W}$ : joint diagonalization

- $J(\mathbf{W}_C|\mathbf{W}_R)$ or $J(\mathbf{W}_R|\mathbf{W}_C) \Rightarrow$ efficiently minimized modified Pham’s technique [Pham 2001]

- Iterative constrained joint diagonalization of $\{\tilde{D}_c\}$ or $\{\tilde{D}_r\}$

- Convergence is proven, generally very fast (4 iterations)
ML estimation of SKP2 covariance matrices: algorithm

**Step 0 Initialisation**

- Compute $\hat{R}_{yy} = \frac{1}{L} \sum_{l=1}^{L} y_{PS}(l)y_{PS}^\dagger(l)$, set $W_C(0), W_R(0), n = 0$
ML estimation of SKP2 covariance matrices: algorithm

**Step 0  Initialisation**

- Compute $\hat{R}_{yy} = \frac{1}{L} \sum_{l=1}^{L} y_{PS}(l)y_{PS}^\dagger(l)$, set $W_C(0), W_R(0), n = 0$

**Step 1  Diagonalization**

- Compute $\tilde{D}(n+1) = W(n)\hat{R}_{yy}W^\dagger(n), n = n + 1$
ML estimation of SKP2 covariance matrices: algorithm

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- Compute $\hat{R}_{yy} = \frac{1}{L} \sum_{l=1}^{L} y_{PS}(l)y_{PS}^\dagger(l)$, set $W_C(0), W_R(0), n = 0$

**Step 1 Diagonalization**

- Compute $\tilde{D}(n+1) = W(n)\hat{R}_{yy}W^\dagger(n), n = n+1$

**Step 2 Minimization using Pham’s technique**

- $W_C(n) = \arg\min_{W_C} J(W_C|W_R(n-1)), W_R(n) = \arg\min_{W_R} J(W_R|W_C(n-1))$
ML estimation of SKP2 covariance matrices : algorithm

**Step 0** Initialisation

- Compute $\hat{R}_{yy} = \frac{1}{L} \sum_{l=1}^{L} y_{PS}(l)y_{PS}^\dagger(l)$, set $W_C(0), W_R(0), n = 0$

**Step 1** Diagonalization

- Compute $\tilde{D}(n+1) = W(n)\hat{R}_{yy}W^\dagger(n), n = n + 1$

**Step 2** Minimization using Pham’s technique

- $W_C(n) = \arg \min_{W_C} J(W_C|W_R(n-1)), W_R(n) = \arg \min_{W_R} J(W_R|W_C(n-1))$

**Step 3** Convergence?

- If no : Go to Step 1

**Step 4** Identification

- $\tilde{D} = \text{diag}(\hat{W}R_{yy}\hat{W}^\dagger), \hat{R}_{PS-ML} = \hat{W}^{-1}\tilde{D}\hat{W}^{-\dagger}$
- From $\hat{R}_{PS-ML}$, identify $\hat{C}_g, \hat{R}_g, \hat{C}_v, \hat{R}_v$
Global relative rms error

\[ \text{rmse} = \frac{E \left( \left\| \hat{R}_{PS} - R_{PS} \right\|^2_F \right)}{\left\| R_{PS} \right\|^2_F}, \quad \text{with} \quad \hat{R}_{PS} = \hat{R}_{yy}, \hat{R}_{PS-LS}, \hat{R}_{PS-ML} \]
Coherence locations in the complex plane

\[ N_p = 3, \; N_s = 5, \; \text{Ground: rank}(C_g) = 2, \; \text{Volume: full rank, minimum phase} \]

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MB-RVOG, ML SKP-2, PolinSAR 2013, Frascati
Coherence wise rms error

$$\text{rmse}(\gamma_{ij}) = E(||\gamma_{ij} - \hat{\gamma}_{ij}||^2)$$

Equal performance for low rank (and DoF) $R_g$, well chosen solution.
Polarimetric rms error

\[
\text{rmse}_i = \frac{\mathbb{E} \left| \mathbf{C}_{PSi} - \hat{\mathbf{C}}_{PSi} \right|^2}{\left| \mathbf{C}_{PSi} \right|^2_F}
\]

Large LS error over low rank \( \mathbf{C}_g \)
MB-RVOG model validity vs estimation

\[
\frac{\text{numb. unfeasible exp.}}{\text{numb. exp.}} \times 100
\]

![Graph showing MB-RVOG model validity vs estimation for different values of L. The graph compares the number of unfeasible experiments to the total number of experiments for both LS and ML methods. The LS line shows a steady decrease, while the ML line fluctuates initially before stabilizing.](image-url)
Tomography at P band of a TROPISAR profile (ONERA/SETHI)

\[ N_p = 3, \quad N_s = 6 \quad L = 16 < N_s N_p \]
$N_p = 3, N_s = 6 \quad L = 16 < N_s N_p$

Unfeasible Points
43%

Unfeasible Points
6%
Conclusion

Numerical ML estimation of the MB-RVOG model

- Rigorous and fast
- Adapted to POL-inSAR and TOMO-SAR
- Could also be used in medical imaging, image processing . . .

Performance

- Fewer looks needed
- Significant variance reduction $LS \rightarrow ML \equiv$ unformated $\rightarrow LS$
- Better estimation of polarimetric features, not only spatial ones