The edgelist phase unwrapping algorithm and its application to time-series InSAR

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Outline

• Minimum Cost Flow (MCF) unwrapping.
• Edgelist unwrapping formulation.
• Properties of formulation
  – Equivalence to MCF
  – Total unimodularity
  – GPS data as constraints
• Application to Central San Andreas Fault
MCF unwrapping

• Costantini (1998); Costantini and Rosen (1999)
• Residues modeled as a source or sink.
• Flow associated with every edge of the grid.

\[
\begin{array}{c}
0 \\
K_1 \\
K_3 \\
K_2 \\
\text{Residue } = -1
\end{array}
\quad
\begin{array}{c}
0 \\
K_1 \\
K_3 \\
K_2 \\
\text{Residue } = 1
\end{array}
\]

\[
\begin{align*}
\frac{4\pi}{3} & \\
\frac{2\pi}{3} & \\
\frac{4\pi}{3} & \\
\frac{2\pi}{3} & \\
\end{align*}
\]
• Each closed loop represents a constraint.
• A cost is associated with flow on each edge.
• Formulation is an integer program.
• **Advantage:** Total unimodularity
• LP solvers to solve minimum $L_1$ formulation.
• Extended for SBAS by Pepe and Lanari (2005).
• Solution dependent on cost functions.
Edgelist unwrapping formulation

- Dual formulation of the MCF.
- **Building block:** Edges as constraints.
- Formulation is still an integer program.
- Control variable for every edge as well as vertex.

\[
K_{ab} = \phi_b - \phi_a + \left[ \frac{\psi_b - \psi_a}{2\pi} \right] = K_{ab}
\]

\[
\phi_b = \psi_b + 2\pi \cdot n_b
\]

- \(K_{ab}: \) Flow variables
- \(n_a: \) Node potentials
Equivalence of MCF and edgelist

**Edgelist Formulation**

\[
\begin{align*}
K_{ab} &= n_b - n_a + \left[ \frac{\psi_b - \psi_a}{2\pi} \right] \\
K_{bc} &= n_c - n_b + \left[ \frac{\psi_c - \psi_b}{2\pi} \right] \\
K_{ca} &= n_a - n_c + \left[ \frac{\psi_a - \psi_c}{2\pi} \right]
\end{align*}
\]

**MCF Formulation**

\[
K_{ab} + K_{bc} + K_{ca} = \\
\left[ \frac{\psi_b - \psi_a}{2\pi} \right] + \left[ \frac{\psi_c - \psi_b}{2\pi} \right] + \left[ \frac{\psi_a - \psi_c}{2\pi} \right]
\]
The unwrapping grid

• 2D Delaunay triangulation of spatial edges.
• Rectangular facets including temporal edges.
• Could used 3D Tessellation alternatively.
• Connected grid with no leaf nodes.
• All edges should be part of a loop.
Customized unwrapping grids

**Original Triangulation**
- Triangles with long edges.
- Edges across water bodies.
- Adjusting cost functions is hard.

**Modified Triangulation**
- Fewer long edges.
- No edges across the bay.
- Realistic cost functions.
## Number of variables

Formulating a regularly sampled 2D interferogram.

<table>
<thead>
<tr>
<th></th>
<th>MCF</th>
<th>Edgelist</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of variables</td>
<td>4MN-2M-2N</td>
<td>5MN-2M-2N</td>
</tr>
<tr>
<td>Number of constraints</td>
<td>MN-M-N+1</td>
<td>2MN-M-N</td>
</tr>
<tr>
<td>Number of sparse matrix entries</td>
<td>8MN-8M-8N+8</td>
<td>8MN-4M-4N</td>
</tr>
</tbody>
</table>

- 2D problem -> 25% increase in variables and 100% increase in constraints.
- 3D problem -> up to 20% increase in variables and number of constraints are more comparable.
- Easy to construct the sparse constraint matrix.
The constraint matrix structure

\[ n_b - n_a + \begin{bmatrix} \frac{\psi_b - \psi_a}{2\pi} \end{bmatrix} = K_{ab} \]

\[ n_b - n_a + P_{ab} - Q_{ab} = \begin{bmatrix} \frac{\psi_b - \psi_a}{2\pi} \end{bmatrix} \]

\[ \begin{bmatrix} N \\ P \\ Q \end{bmatrix} = A \begin{bmatrix} I \\ -I \end{bmatrix} \cdot \begin{bmatrix} N \\ P \end{bmatrix} = b \]

• A is the node incidence matrix.
• Directionality due to labeling of edges (i,j) s.t. i < j.
• \([A \mid I \mid -I]\) also has (+1,0,-1) structure.
• All equality constraints. No slack variables.
Total Unimodularity (TUM)

- Theorem 1: A matrix with utmost one +1 and one -1 in each column is totally unimodular. (Node incidence matrix)

- Theorem 2: If A is totally unimodular, -A, A^T, [A|I] are also totally unimodular.

- Theorem 3: When constraints are of form Ax=b, where A is totally unimodular and b is all integers; all the basic feasible solutions are integral and there exists an optimal integral solution.

Hence, the linear programming relaxation exactly solves the integer program as well. Allows us to use LP solvers.

(MCF also has this property).
Example problem

**Initial Solution**
- Set all node potentials to zero.
- Set $P_{25}, P_{45}$ to +1.
- Set $Q_{56}, Q_{58}$ to -1.
- Total cost of “4”.

**Optimal Solution**
- Set all node potentials to zero.
- Set $n_5$ to +1.
- Same solution as MCF.
- Total cost of “0”.
Why edgelist formulation?

• What do we gain over MCF?
• Edgelist allows us to incorporate other constraints.
  • GPS observations.
  • Leveling data, creepmeter etc.
• Controls over the edges as well as nodes.
• Constraints on temporal edges as well as spatial edges.
• Can use skeleton of GPS observations to guide phase unwrapping of time-series data.
• Can also use partially coherent pixels in the unwrapping process.
GPS data and MCF formulation

\[ \sum_{p_i} \psi_i - \psi_j \frac{2\pi}{2\pi} = \Delta N_{gps} \]

Integrated sum of the unwrapped phase differences along the edges of any path should correspond to the GPS measurement.

Violates the TUM (+1,-1,0) structure of the constraint matrix and hence, cannot be reduced to a simple Linear Program.
GPS data and edgelist formulation

\[ n_P - n_Q + \left[ \frac{\psi_P - \psi_Q}{2\pi} \right] = \Delta N_{gps} \]

- Introduce a new edge between GPS stations.
- Node incidence matrix still has TUM structure.
- We exploit the fact that edgelist does not distinguish between 2D and 3D data.

Needs to be further tested on real data sets in areas with dense GPS networks.
Example: San Andreas Fault

Courtesy: Isabelle Ryder (Berkeley)
Problems with unwrapping

- Baseline less than 300 m.
- Couple of time gaps of more than a year.
- Fault cuts right through the image.
- No zero gradient area.
- Large temporal baseline gaps result in unwrapping errors.
- Residues not compensated on the fault.
Unwrapping model

- USGS fault trace used to identify edges cutting across fault.
- 1 Km zone around the fault used to subsidize edge costs.
- Slice of zero phase for reference interferogram.
- Node potentials for reference slice constrained to zero.
- 3D phase unwrapping.
- Assumption: Atmospheric effects have been neglected.
Results

- Step function across the fault is clearly visible.
- The velocities exhibit a spatial variation of 1 mm/yr LOS.
- The creep rate is roughly 7 mm/yr LOS or 22 mm/yr.
- Estimates from other methods vary from 18 – 35 mm/yr.
Results

- The average velocities over a long time-period are consistent.
- Non-linear creep.
- Non-uniform creep rate.
- Possibility of vertical deformation.
Conclusions

• Edgelist formulation allows for greater control.
• Allows for wider range of constraints.
• Allows inclusion of GPS measurements and other observations in the phase unwrapping process.
• Even if GPS data is unavailable, a priori information regarding the region of interest can be utilized.
• Can be adopted for space-time MCF as well.
• Can be easily implemented using LP solvers.

Future Work

• Testing of algorithm with dense GPS network.
• Identify limitations with real data and effect of over-constraining the solutions.