Upward continuation of satellite altimeter data for GOCE validation

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1 **Introduction**
- Satellite altimetry

2 **Upward experiment**
- Global approximation
- Input signal

3 **Numerical comparisons**
- Upwarded 0th derivative: T
- Upwarded 1st derivative: $T_r, T_{\bar{z}}$
- Upwarded 2nd derivative: $T_{rr}, T_{\bar{z}\bar{z}}$

4 **Concluding remarks**

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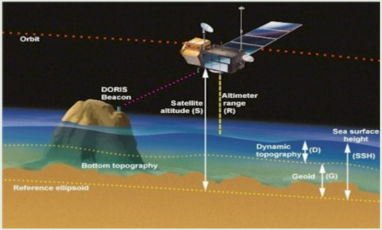
Roughly on satellite altimetry

- SA gives the geoid undulation since it holds $MSS = DOT + N$ (MSS-mean sea surface, DOT-dynamic ocean topography).
- Having DOT at hand we can determine disturbing/anomalous potential from the Bruns formula $T = N\gamma$.

SA - principle (from AVISO-CNES)



Some assumptions

- DOT is also unknown $\Rightarrow$ not solved here
- SA coverage isn't global $\Rightarrow$ not considered
- $\Rightarrow$ T globally is our input for the base functions experiments

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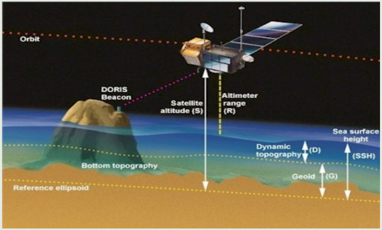

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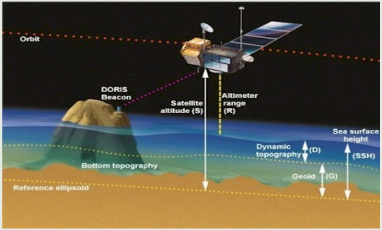

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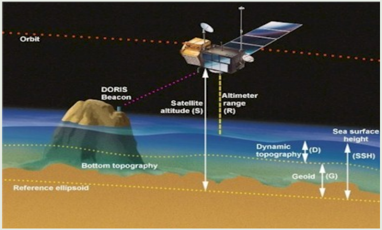

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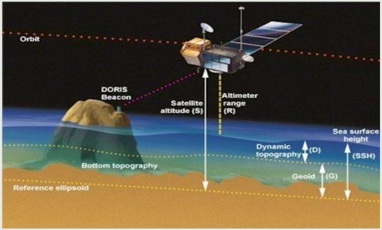

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Numerical test of the approximation

1. Start with a signal on the geoid - T
2. Use both kinds of approximation $\Rightarrow \{C^e_{nm}, S^e_{nm}\}$ and $\{C^s_{nm}, S^s_{nm}\}$
3. Map both sets by upward operators onto the potential functionals at satellite altitude and compare EHS and SHS

Grid settings

- Regular grid on **sphere** (geocentric co-latitude θ) is not regular on the **ellipsoid** (reduced co-latitude ϑ)
- Trade-off $\Rightarrow$ mixture of both
- For SHA $P \in \{r, \theta + \theta(\vartheta), \lambda\}$
- For EHA $P \in \{u, \vartheta(\theta) + \vartheta, \lambda\}$

Harmonic analysis

- $\Rightarrow$ "Semi-regular" grid $f = (2N_{max} - 1, 2N_{max})$
- $\Rightarrow$ WLS solution for blocks used
- $\Rightarrow$ Latitudinal weights $W_i(\theta) = 2\frac{\sin\theta_i}{\sum_{i=1}^{2N-1}\sin\theta_i}$

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Spherical harmonics

very well represent a functional $f = f(r, \theta, \lambda)$ on Earth $\backsim$ spherical approximation

$$T^s = \frac{GM}{a} \sum_{n,m} \left(\frac{a}{r}\right)^{n+1} (C^s_{nm} \cos m\lambda + S^s_{nm} \sin m\lambda) P_{nm}(\cos\theta) \qquad (1)$$

Ellipsoidal harmonics

are much closer to Earth's geometry, functional $f = f(u, \vartheta, \lambda)$

$$T^e = \frac{GM}{a} \sum_{n,m} \frac{Q_{nm}(\frac{u}{E})}{Q_{nm}(\frac{b}{E})} (C^e_{nm} \cos m\lambda + S^e_{nm} \sin m\lambda) P_{nm}(\cos\vartheta) \qquad (2)$$

or with Jekeli's renormalization $S_{nm}(\frac{u}{E})/S_{nm}(\frac{b}{E}) = Q_{nm}(\frac{u}{E})/Q_{nm}(\frac{b}{E})$

"Normal" derivatives (z axis in LNOF)

$$T^s_r = \frac{\partial V^s}{\partial r} \approx \frac{\partial V^e}{\partial \bar{z}} = T^e_{\bar{z}}$$

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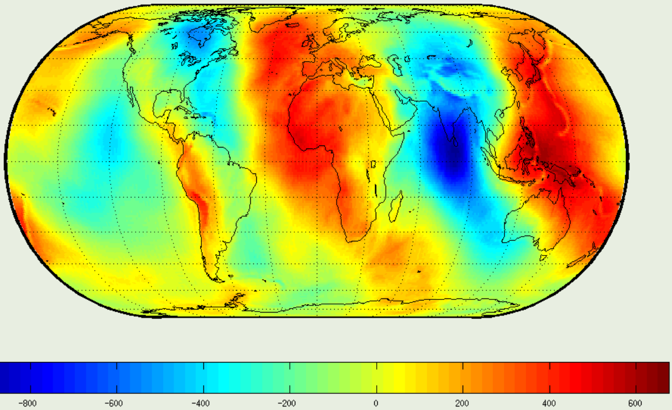
Spherical and ellipsoidal harmonic analysis of T on the geoid

- Disturbing potential on the geoid, ITG03 model, $N_{max} = 180$
- Ellipsoidal and spherical analysis $N_{max} = 180$

T^s from SHS, $[m^2 \cdot s^{-2}]$



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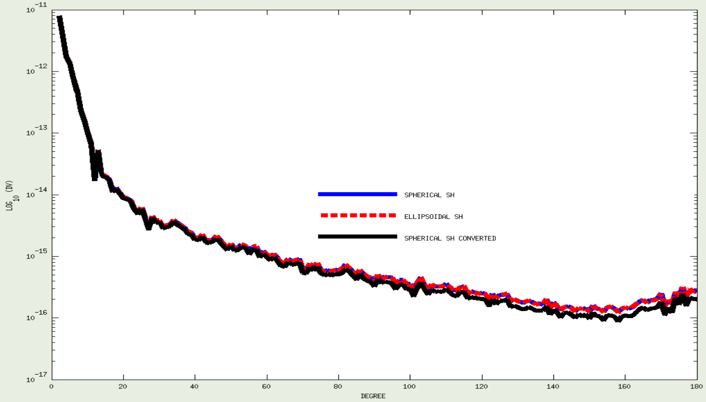
Degree variances of the derived coefficients



Synthesis of T^s, T^e, T^s_{conv}, $u = b + 255$ km, $N_{max} = 180$

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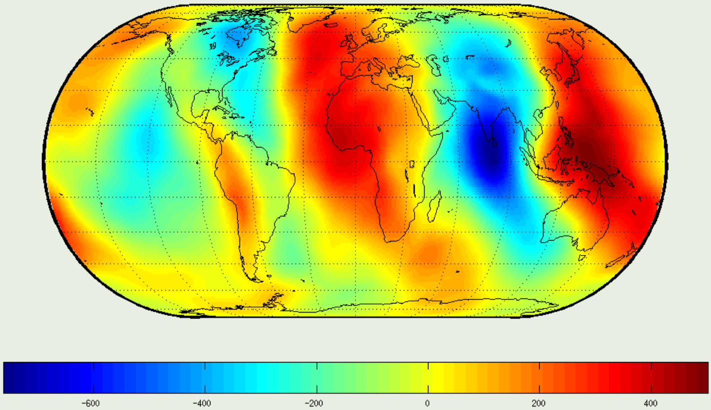

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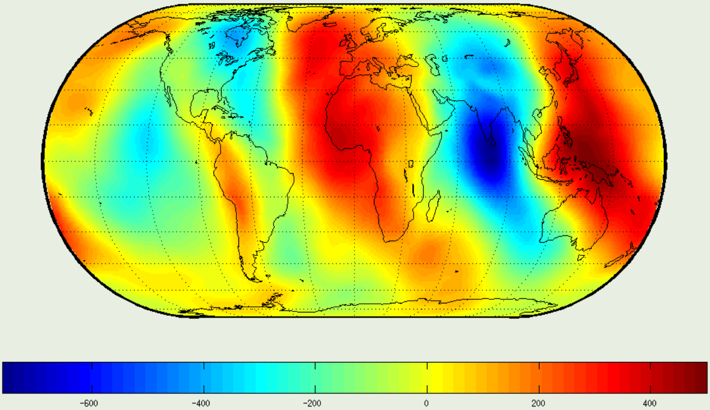
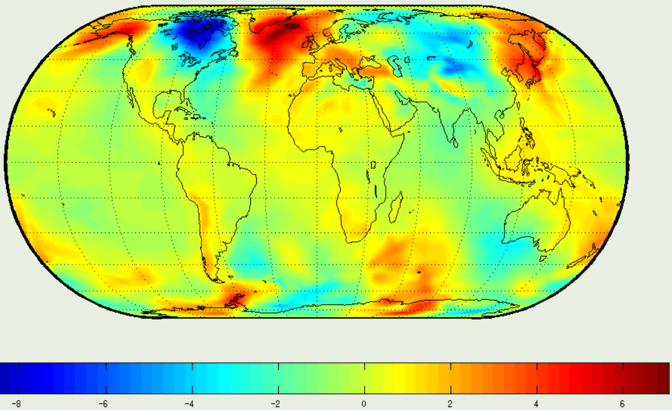
T^e from EHS, $[m^2 \cdot s^{-2}]$



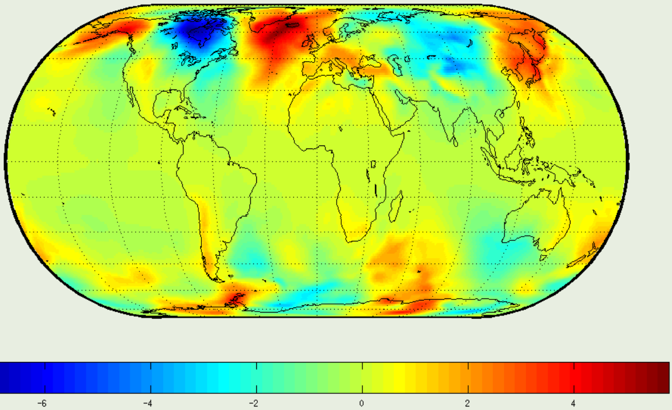
Synthesis of T^s, T^e, T^s_{conv}, $u = b + 255$ km, $N_{max} = 180$

$T^s - T^e$, RMS $= 1.74\ m^2 \cdot s^{-2}$



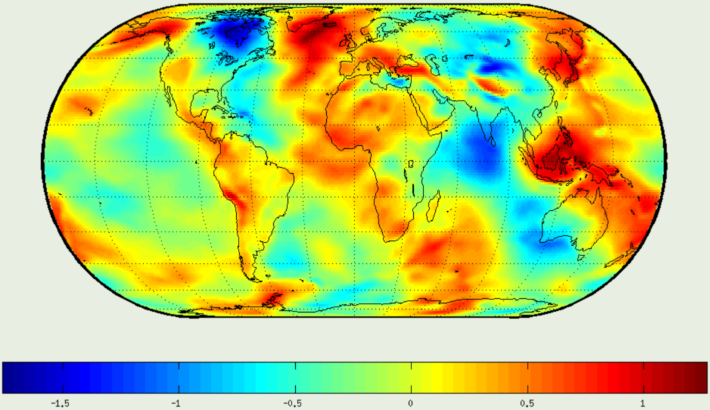
Synthesis of T^s, T^e, T^s_{conv}, $u = b + 255$ km, $N_{max} = 180$

$T^s - T^s_{conv}$, RMS $= 1.37\ m^2 \cdot s^{-2}$



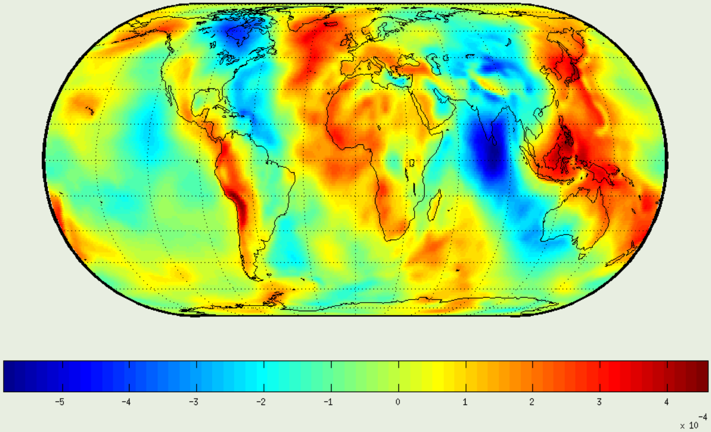
Synthesis of T^s, T^e, T^s_{conv}, $u = b + 255$ km, $N_{max} = 180$

$T^s_{conv} - T^e$, RMS $= 0.41\ m^2 \cdot s^{-2}$



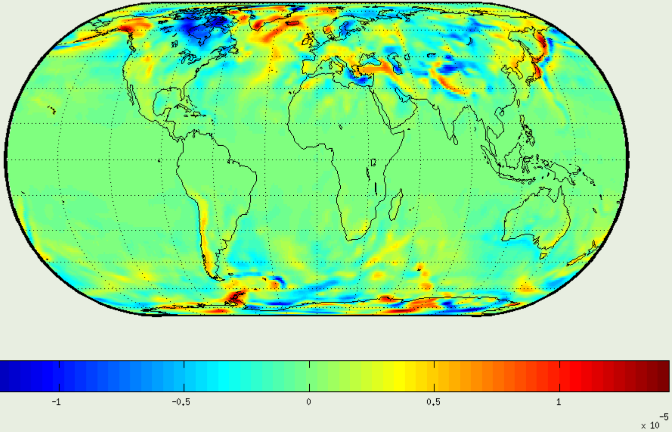
Synthesis of T_r, $T_{\bar{z}}$, $u = b + 255$ km,$N_{max} = 180$

T_r^s from SHS, $[m \cdot s^{-2}]$



Synthesis of T_r, $T_{\bar{z}}$, $u = b + 255$ km,$N_{max} = 180$

$T_r^s - T_{\bar{z}}^e$, RMS $= 0.253\ mGal$



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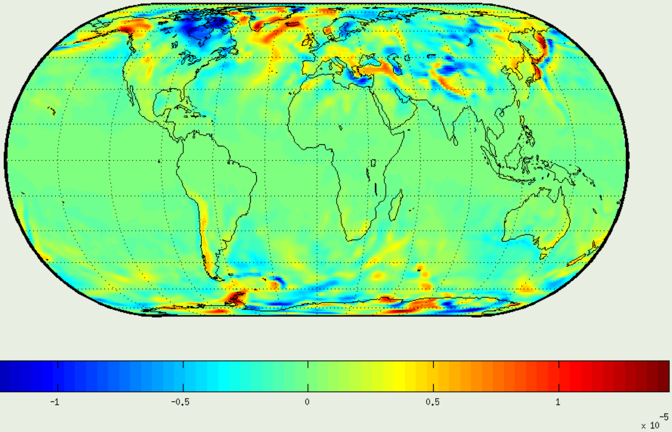
$T_r^s - T_{r,conv}^s$, RMS $= 0.255\ mGal$



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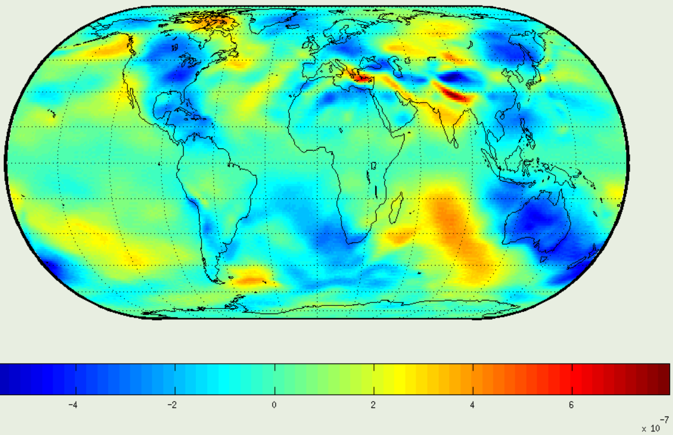
$T^s_{r,conv} - T^e_{\bar{z}}$, RMS $= 0.014\ mGal$



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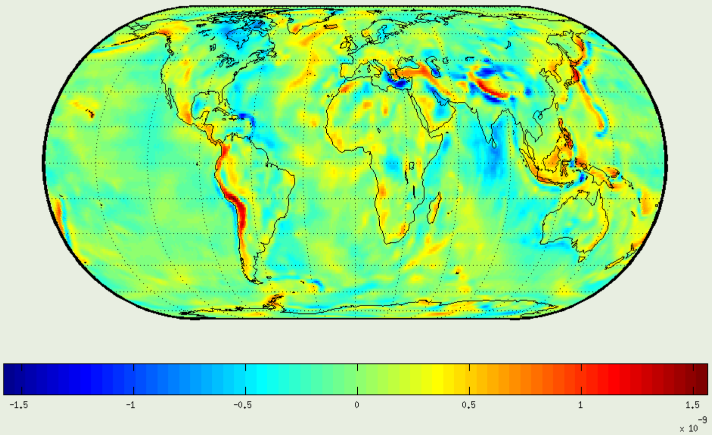
T^s_{rr} from SHS, $[s^{-2}]$



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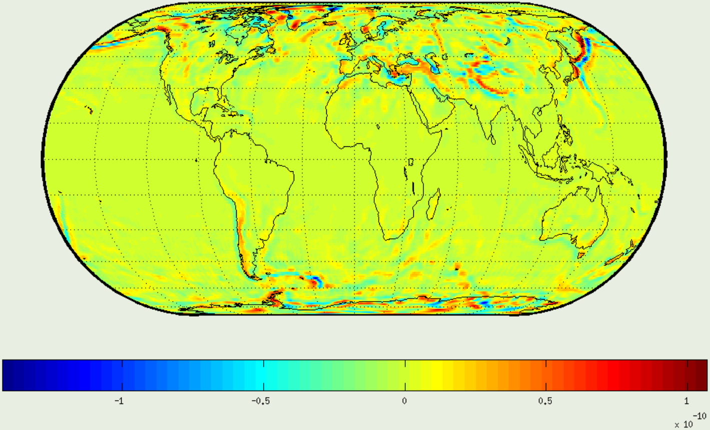
$T^s_{rr} - T^e_{\bar{z}\bar{z}}$, RMS $= 15.47\ mE$



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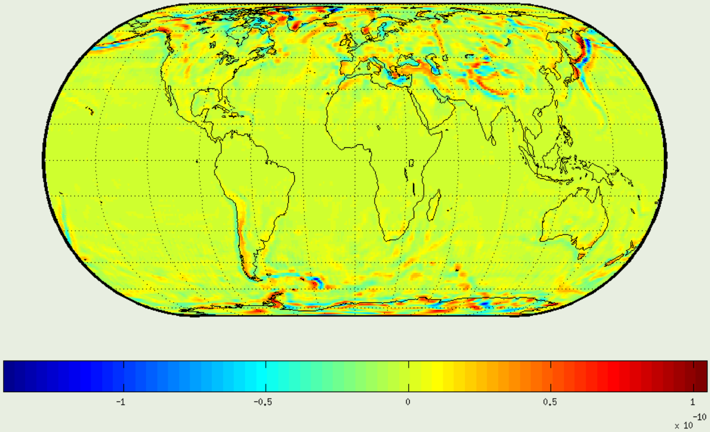
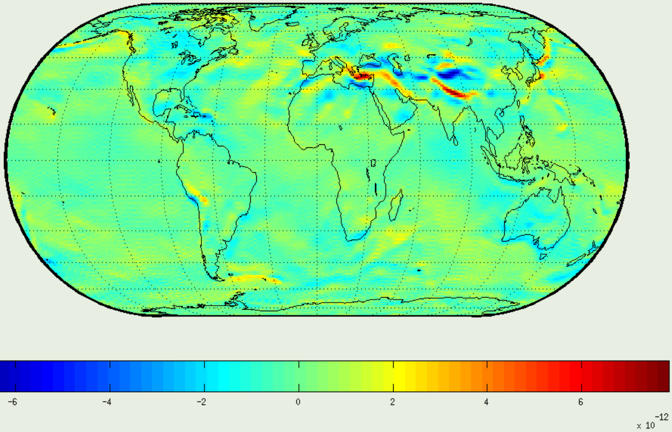
$T^s_{rr} - T^s_{rr,conv}$, RMS $= 15.44\ mE$



Synthesis of T_{rr}, $T_{\bar{z}\bar{z}}$ at $u = b + 255$ km

$T^{s}_{rr,conv} - T^{e}_{\bar{z}\bar{z}}$, RMS $= 0.69\ mE$



Concluding remarks

RMS of diffs.	$T\ [m^2 \cdot s^{-2}]$	$T_r, T_{\bar{z}}\ [mGal]$	$T_{rr}, T_{\bar{z}\bar{z}}\ [mE]$
Sph - Ell	1.74	0.253	15.5
Sph - Sph(conv.)	1.37	0.255	15.4
Ell - Sph(conv.)	0.41	0.014	0.7

- We have compared three sets of coefficients coming from one input (2x spherical and 1x ellipsoidal) via harmonic synthesis on the $u = b + 255$ km for three orders of derivative of T.
- Good agreement achieved when SHS with converted coefficients and EHS were used
- $\Longrightarrow$ when validation uses the global approximation of the ground data, EH and SH(converted) "suit" more to this task
- $\Longrightarrow$ global gravity field models based on the ellipsoidal analysis might have principal advantages (e.g. EGM08)

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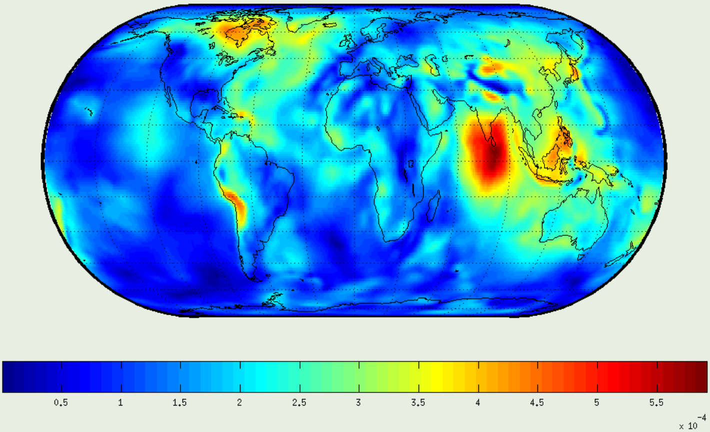
Thank you!

Synthesis of $|\nabla T^s|$, $|\nabla T^e|$, $u = b + 255$ km,$N_{max} = 180$

$$
\begin{aligned}
|\nabla T^s|^2 &= \left(\frac{\partial T}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial T}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial T}{\partial \lambda}\right)^2 \\
|\nabla T^e|^2 &= \left(\frac{1}{w}\frac{\partial T}{\partial u}\right)^2 + \left(\frac{1}{w\sqrt{u^2+E^2}}\frac{\partial T}{\partial \vartheta}\right)^2 + \left(\frac{1}{\sqrt{u^2+E^2}\sin\vartheta}\frac{\partial T}{\partial \lambda}\right)^2 \\
w &= \sqrt{\frac{u^2+E^2\cos^2\vartheta}{u^2+E^2}}
\end{aligned}
$$

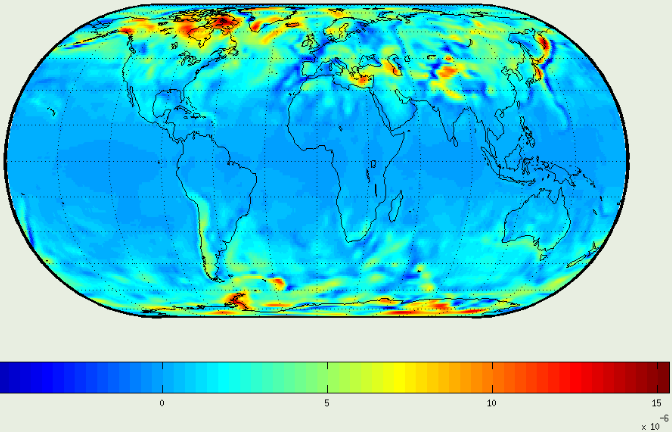
Synthesis of $|\nabla T^s|$, $|\nabla T^e|$, $u = b + 255$ km, $N_{max} = 180$

$|\nabla T^s|$ from SHS, $[m \cdot s^{-2}]$



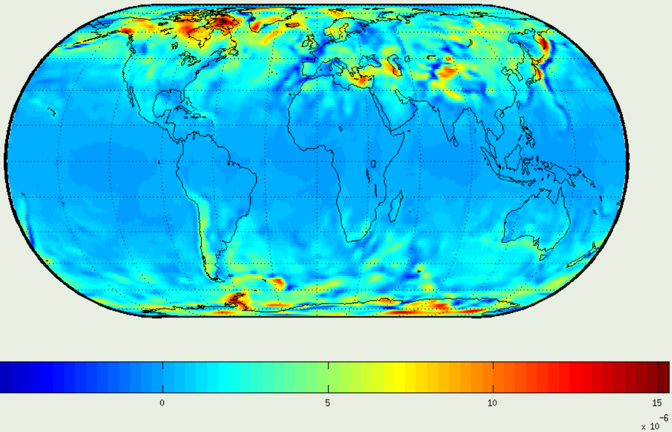
Synthesis of $|\nabla T^s|$, $|\nabla T^e|$, $u = b + 255$ km, $N_{max} = 180$

$|\nabla T^s| - |\nabla T^e|$, RMS $= 0.307\ mGal$



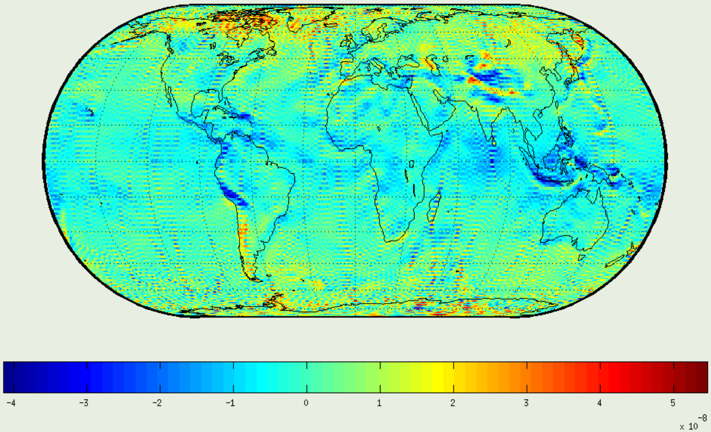
Synthesis of $|\nabla T^s|$, $|\nabla T^e|$, $u = b + 255$ km, $N_{max} = 180$

$|\nabla T^s| - |\nabla T^s_{conv}|$, RMS $= 0.307\ mGal$



Synthesis of $|\nabla T^s|$, $|\nabla T^e|$, $u = b + 255$ km, $N_{max} = 180$

$|\nabla T^s_{conv}| - |\nabla T^e|$, RMS $= 0.001\ mGal$



$$
\begin{aligned}
T_{\bar{z}} &= \frac{GM}{a}\frac{v}{L}\sum_{n,m}\frac{\frac{\partial S_{nm}(\frac{u}{E})}{\partial u}}{S_{nm}(\frac{b}{E})}\left(A_{nm}\cos m\lambda + B_{nm}\sin m\lambda\right)P_{nm}(\cos\vartheta) \\
T_{\bar{z}\bar{z}} &= \frac{GM}{a}\frac{v^2}{L^2}\sum_{n,m}\frac{\frac{\partial^2 S_{nm}(\frac{u}{E})}{\partial u^2}}{S_{nm}(\frac{b}{E})}\left(A_{nm}\cos m\lambda + B_{nm}\sin m\lambda\right)P_{nm}(\cos\vartheta) \\
&- \frac{GM}{a}\frac{uE^2\sin^2\vartheta}{L^4}\sum_{n,m}\frac{\frac{\partial S_{nm}(\frac{u}{E})}{\partial u}}{S_{nm}(\frac{b}{E})}\left(A_{nm}\cos m\lambda + B_{nm}\sin m\lambda\right)P_{nm}(\cos\vartheta) \\
&+ \frac{GM}{a}\frac{E^2\sin\vartheta\cos\vartheta}{L^4}\sum_{n,m}\frac{S_{nm}(\frac{u}{E})}{S_{nm}(\frac{b}{E})}\left(A_{nm}\cos m\lambda + B_{nm}\sin m\lambda\right)\frac{\partial P_{nm}(\cos\vartheta)}{\partial\vartheta} \\
v^2 &= u^2 + E^2 \\
L^2 &= u^2 + E^2\cos^2\vartheta
\end{aligned}
$$