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On the Combination of Terrestrial Data and GOCE Based Models in Earth's Gravity Field Studies: Compatibility and Optimization

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1. Introduction – Integral Kernels

When dealing with the relationship between the *disturbing potential* T and *gravity disturbances* δg, we known that

$$T(x) = R\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\frac{1}{n+1}\delta g_n \,, \quad r = |\boldsymbol{x}|$$

and that for $r = R$

$$T = \frac{R}{4\pi}\int_{\sigma} K(\psi)\,\delta g\, d\sigma \qquad \text{with} \qquad K(\psi) = \sum_{n=0}^{\infty}\frac{2n+1}{n+1}P_n(\cos\psi)$$

which sometimes is called *Hotine-Koch function*.

Naturally, *global gravity field models* allow a modification, e.g.

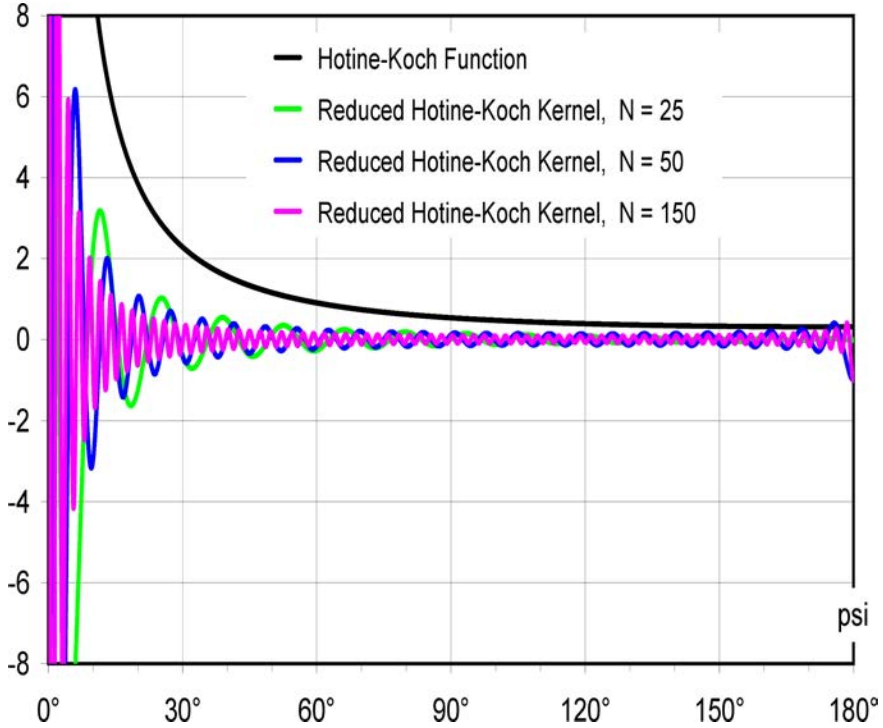
$$T(\boldsymbol{x}) = \frac{R}{r}T_0 + \left(\frac{R}{r}\right)^2 T_1 + \cdots + \left(\frac{R}{r}\right)^N T_{N-1} + R\sum_{n=N}^{\infty}\left(\frac{R}{r}\right)^{n+1}\frac{1}{n+1}\delta g_n$$

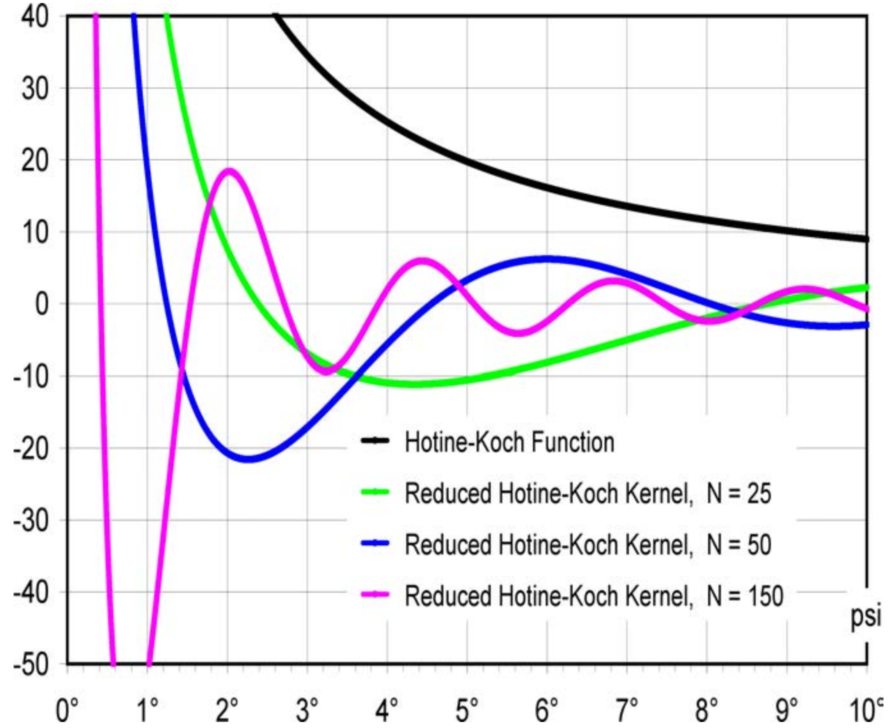
where we compute the disturbances δg with respect to an adopted model, put $T_n = 0$ for $n = 0, 1, 2, \ldots, N-1$ and

subsequently work with the *reduced Hotine-Koch kernel*

$$K_{red}^{(N)}(\psi) = \sum_{n=N}^{\infty} \frac{2n+1}{n+1} P_n(\cos\psi)$$

Graphically the kernel is illustrated in the following figures.

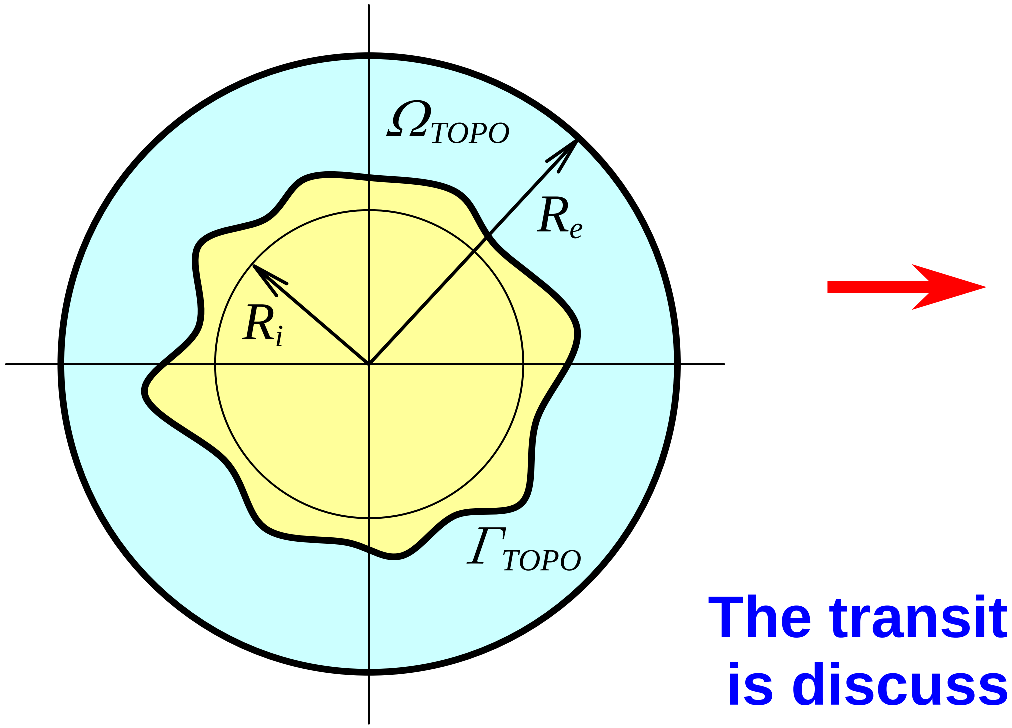


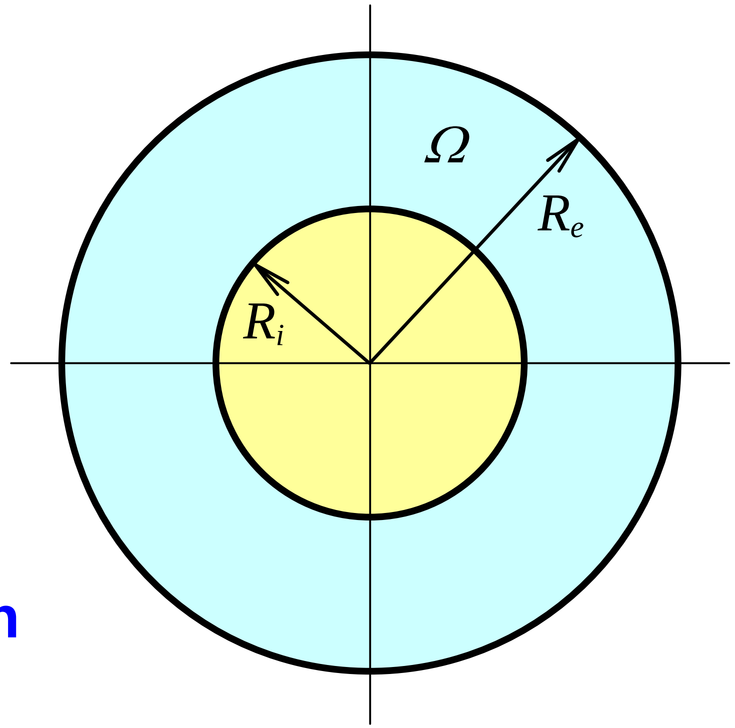


The approach as above is straightforward, but not the only possible.

GOCE **data and** ***terrestrial gravity measurements*** **are** ***two different sources of information*****. Their combination has a tie to** ***potential theory*** **and** ***boundary value problems*****.**

In the sequel Ω **means a solution domain bounded by two surfaces. We can even suppose that** Ω **is bounded by two spheres of radius** R_i **and** R_e**,** $R_i < R_e$**.**





The transition is discussed in ***Holota and Nesvadba (2007)***

2. Boundary Value Problem

If we continue in our considerations, we e.g. can formulate the following problem

$$\Delta T = 0 \quad \textbf{in} \quad \Omega$$

$$\frac{\partial T}{\partial r} = -\delta g \quad \textbf{for} \quad r = R_i \qquad \textbf{and} \qquad T = t \quad \textbf{for} \quad r = R_e$$

where δg **is *the gravity disturbance* and** t **means the *input from an available satellite-only model*.**

The domain Ω **is bounded.** $\Rightarrow$ ***Therefore*, the solution** $T = (r, \varphi, \lambda)$**, we are looking for, *<u>has generally the following form</u>***

$$T = T^{(i)} + T^{(e)} \qquad \textbf{(3)}$$

$$T^{(i)} = \sum_{n=0}^{\infty} \left(\frac{R_i}{r} \right)^{n+1} T_n^{(i)}(\varphi, \lambda) \qquad \textbf{and} \qquad T^{(e)} = \sum_{n=0}^{\infty} \left(\frac{r}{R_e} \right)^{n} T_n^{(e)}(\varphi, \lambda)$$

where $T_n^{(i)}$ **and** $T_n^{(e)}$ **are the *surface spherical harmonics*.**

Using the orthogonality of spherical harmonics, we obtain a ***linear system*** for $T_n^{(i)}$ and $T_n^{(e)}$, that for and individual n ***yields***

$$T_n^{(i)} = \frac{R_i\,\delta g_n + n\,q^n t_n}{D_n} \qquad \textbf{(4a)}$$

and

$$T_n^{(e)} = -\frac{R_i\,q^{n+1}\delta g_n - (n+1)\,t_n}{D_n} \qquad \textbf{(4b)}$$

where

$D_n = n(1+q^{2n+1})+1$ ***is the determinant***, $q = R_i / R_e$

while δg_n and t_n are ***surface spherical harmonics*** in the developments of δg and t , respectively, i.e. in

$\delta g(\varphi,\lambda) = \Sigma_{n=0}^{\infty}\,\delta g_n(\varphi,\lambda)$ and $t(\varphi,\lambda) = \Sigma_{n=0}^{\infty}\,t_n(\varphi,\lambda)$.

3. Compatibility

The solution T is harmonic in Ω. However, the continuation of T for $r > R_e$ *need not be regular at infinity*, i.e., *if analytically extended*, then for $r \to \infty$ it *does not decrease* as c/r (c is a constant) or faster.

This is a *consequence of errors in data.*

- **The data given for $r = R_i$ are enough to determine a harmonic function in $\Omega_{ext} \equiv \{ \boldsymbol{x} \in \boldsymbol{R}^3; r > R_i \}$ and thus in $\Omega \subset \Omega_{ext}$.**
- **The data for $r = R_e$ have the nature of excess data and give rise to the ("*internal*") *term* $(r/R_e)^n T_n^{(e)}$ *not regular at infinity.***

Thus

$$T = T^{(i)} + T^{(e)} , \quad T^{(i)} = \sum_{n=0}^{\infty} \left(\frac{R_i}{r} \right)^{n+1} T_n^{(i)} , \quad T^{(e)} = \sum_{n=0}^{\infty} \left(\frac{r}{R_e} \right)^{n} T_n^{(e)}$$

***is a general solution* in the domain Ω, *but* from the physical point of view its justification rests on a formal basis.**

Nevertheless **the term** $T^{(e)}$ **gives a possibility** ***to confront*** **the** ***two data sources considered*****. To see** ***an example*** **suppose that,**

$$T^{(EGM)} = \sum_{n=0}^{\infty} \left(\frac{R_i}{r} \right)^{n+1} T_n^{(EGM)}(\varphi, \lambda) \qquad \textbf{(9a)}$$

and

$$T^{(GOC)} = \sum_{n=0}^{\infty} \left(\frac{R_i}{r} \right)^{n+1} T_n^{(GOC)}(\varphi, \lambda) \qquad \textbf{(9b)}$$

are the disturbing potentials related to the ***EGM2008*** **and to a** ***GOCE based satellite-only model*****, respectively.**

Subsequently, we can simulate the input ***surface spherical harmonics*** δg_n **and** t_n ***in the following way***

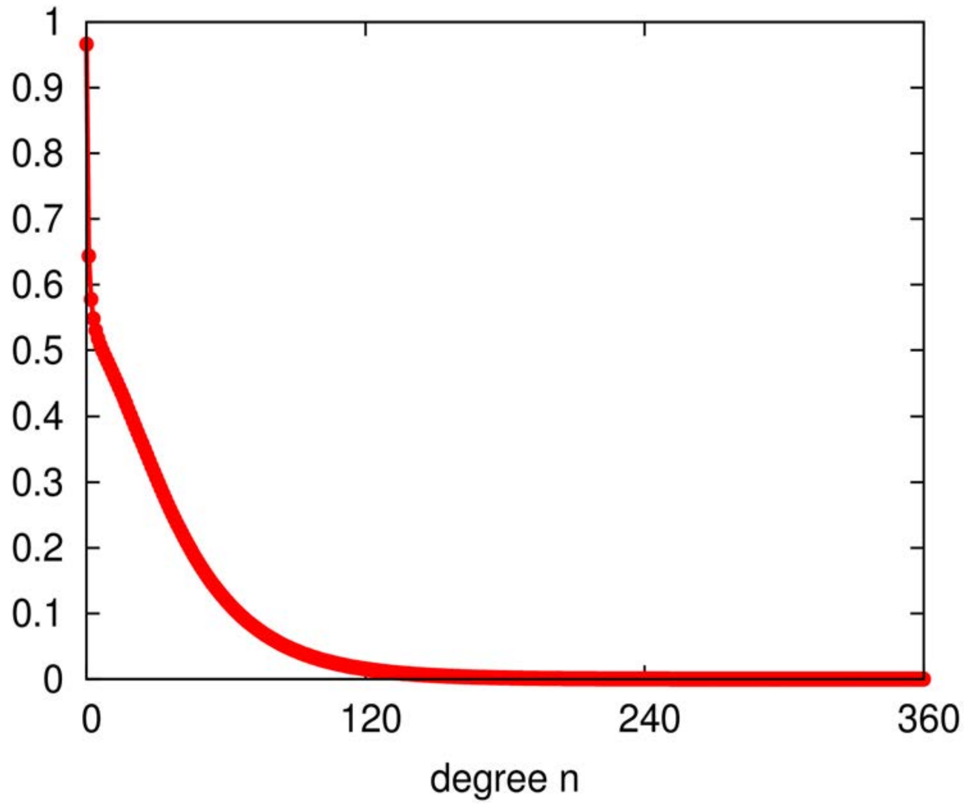
$$\delta g_n = \frac{n+1}{R_i} T_n^{(EGM)} \qquad \textbf{and} \qquad t_n = q^{n+1}\, T_n^{(GOC)}$$

Hence from *Eq.* (*4b*) we get that

$$T_n^{(e)} = c_n \left[T_n^{(GOC)} - T_n^{(EGM)} \right] \quad \textbf{with} \quad c_n = \frac{(n+1)\, q^{n+1}}{n(1+q^{2n+1})+1}$$

and the coefficient c_n is illustrated in the following *Fig. 1*

Figure 1.



A better insight **offers a** ***plot of degree variances*** $\operatorname{var}\{T_n^{(e)}\}$ **.**
Recall, therefore, that

$$T_n^{(EGM)}(\varphi,\lambda)=\sum_{m=0}^{n}\left[\delta\overline{C}_{nm}^{(EGN)}\cos m\lambda+\delta\overline{S}_{nm}^{(EGN)}\sin m\lambda\right]\overline{P}_{nm}(\sin\varphi)$$

and

$$T_n^{(GOC)}(\varphi,\lambda)=\sum_{m=0}^{n}\left[\delta\overline{C}_{nm}^{(GOC)}\cos m\lambda+\delta\overline{S}_{nm}^{(GOC)}\sin m\lambda\right]\overline{P}_{nm}(\sin\varphi)$$

where, $\delta\overline{C}_{nm}^{(EGN)}$, $\delta\overline{S}_{nm}^{(EGN)}$ **and** $\delta\overline{C}_{nm}^{(GOC)}$, $\delta\overline{S}_{nm}^{(GOC)}$ **are coefficients of fully normalized surface spherical harmonics. Hence**

$$\operatorname{var}\left\{T_n^{(e)}\right\}=M\left\{\left[T_n^{(e)}\right]^2\right\}=$$

$$=\left[c_n\right]^2\sum_{m=0}^{n}\left\{\left[\delta\overline{C}_{nm}^{(GOC)}-\delta\overline{C}_{nm}^{(EGM)}\right]^2+\left[\delta\overline{S}_{nm}^{(GOC)}-\delta\overline{S}_{nm}^{(EGM)}\right]^2\right\}$$

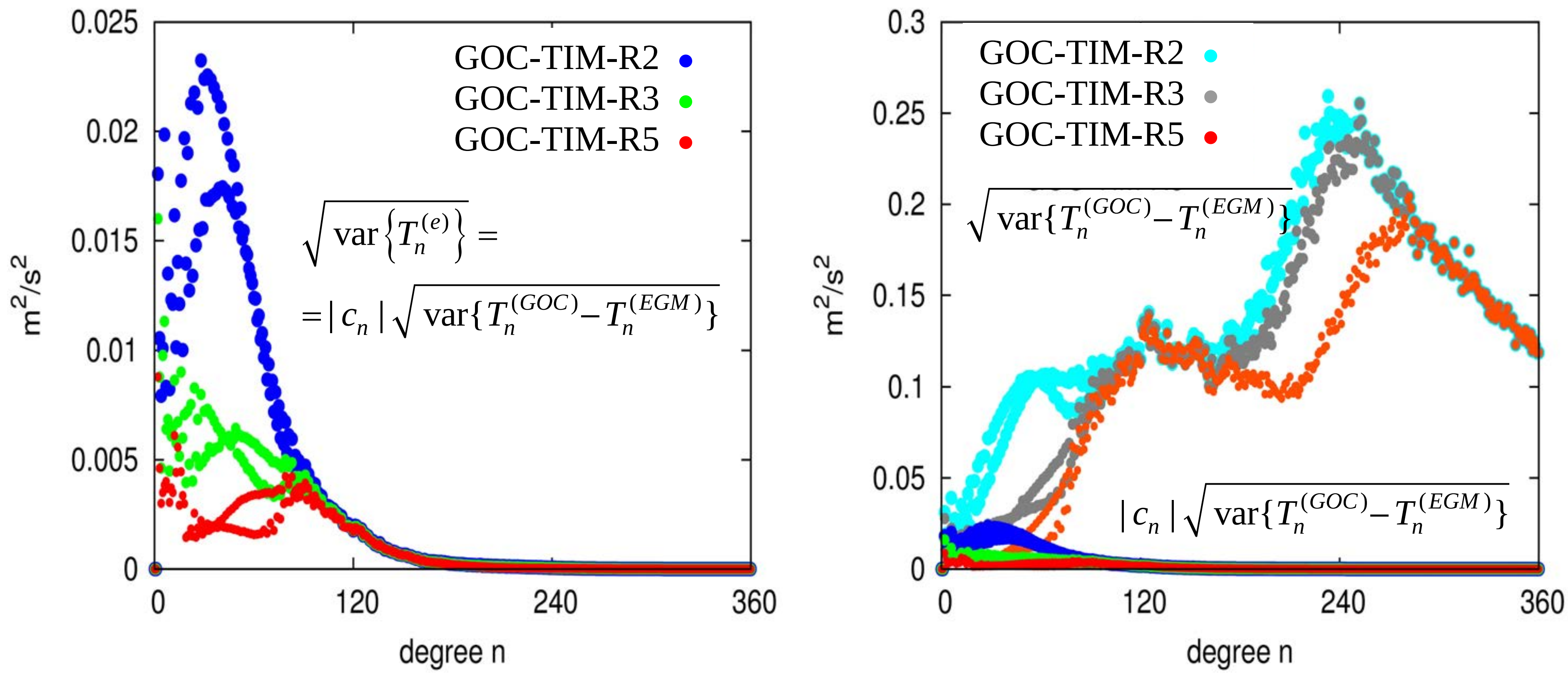
Here M **stands for** ***the average over the whole unit sphere*** **.**

Figure 2. **Two parts of this figure now show:**

(*Left*) **the diagram of** $\sqrt{\mathrm{var}\{T_n^{(e)}\}} = |c_n| \sqrt{\mathrm{var}\{T_n^{(GOC)} - T_n^{(EGM)}\}}$

(*Right*) **the diagram of** $\sqrt{\mathrm{var}\{T_n^{(GOC)} - T_n^{(EGM)}\}}$ **- for comparison**

computed for *3 subsequent GOCE gravity field solutions (TIM)*



We also add a global charts of $T^{(e)}$ and of $T^{(GOC)} - T^{(EGM)}$ for ***gravimetry* (*EGM 2008*)** and ***GO_CONS_GCF_2_TIM_R5*** model.

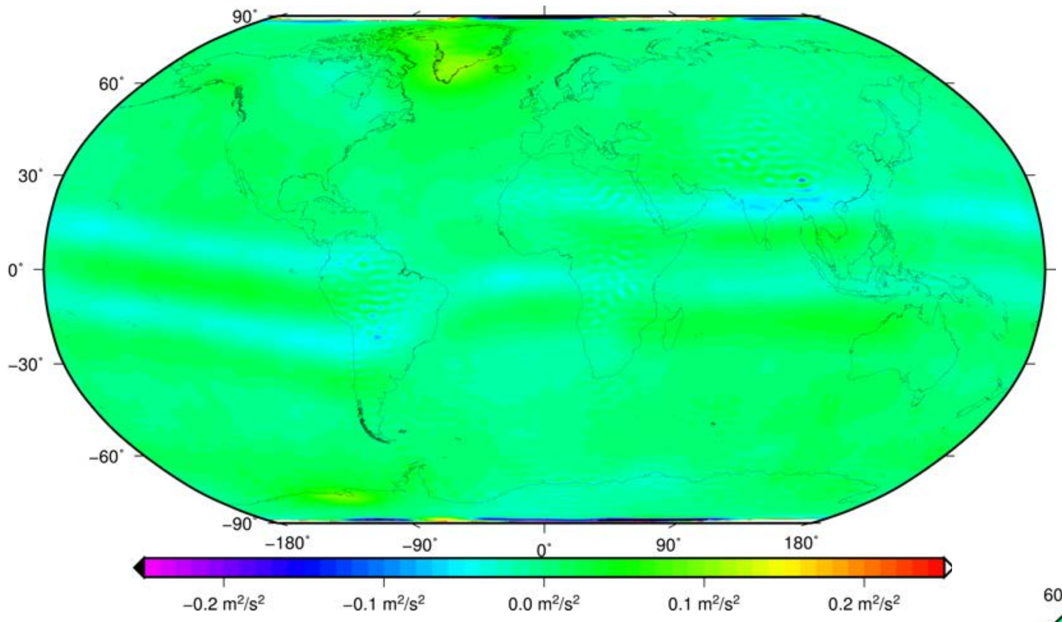


Figure 3a. $T^{(e)}$ for $r = R_i$

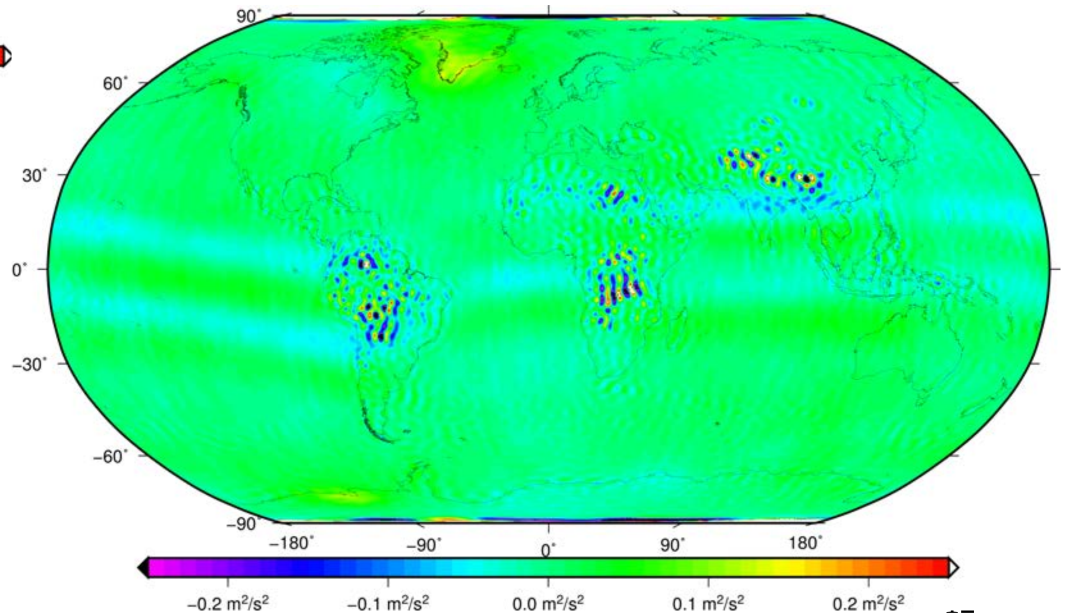


Figure 3b. $T^{(e)}$ for $r = R_e$.

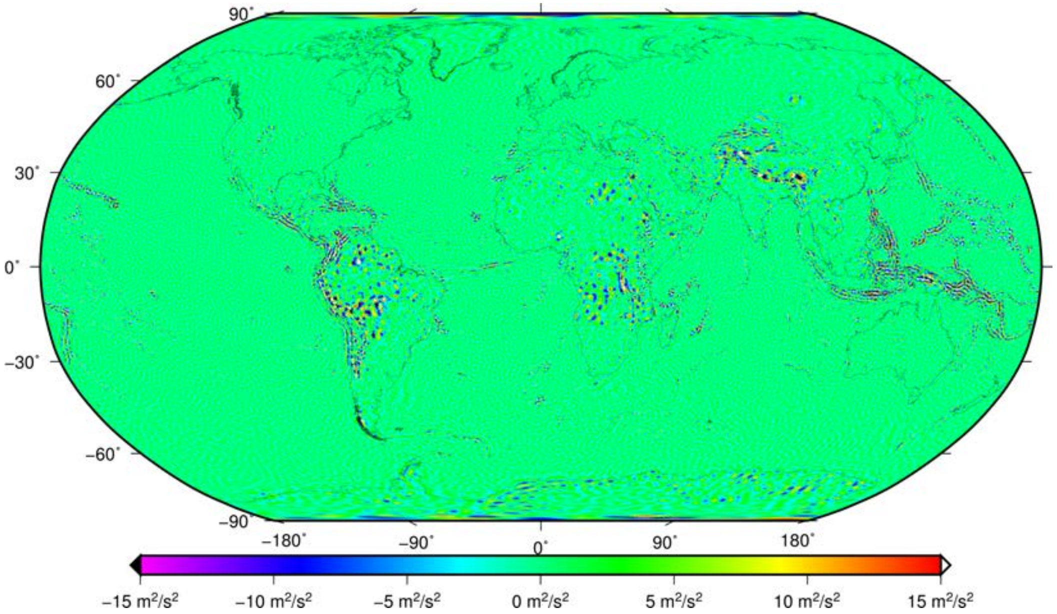


Figure 4a.
$T^{(GOC)} - T^{(EGM)}$ **for** $r = R_i$

Here the scale is 75 times as large.

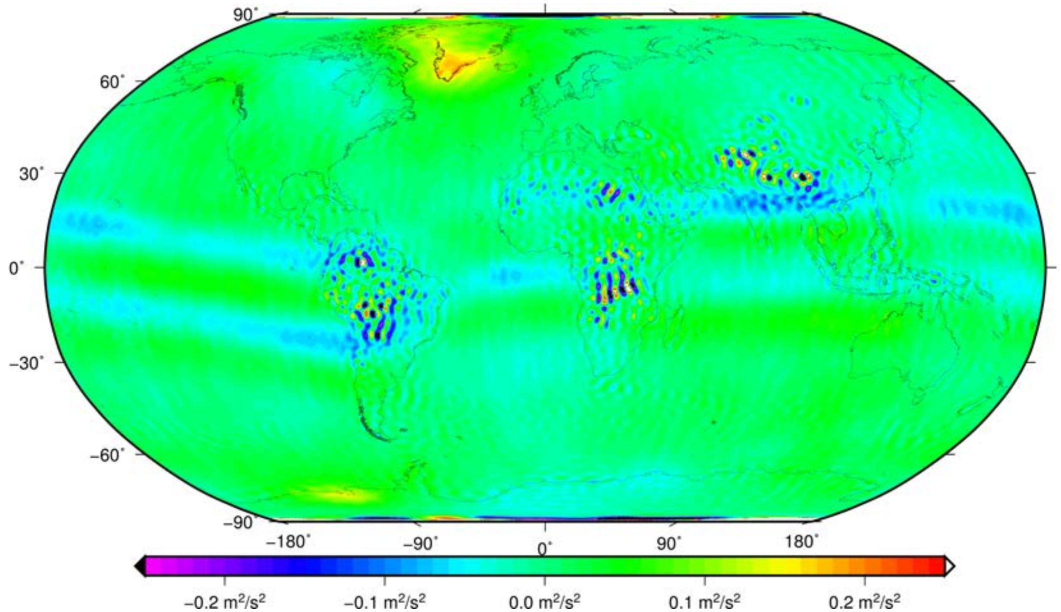


Figure 4b.
$T^{(GOC)} - T^{(EGM)}$ **for** $r = R_e$.

4. Optimization

In solving the incompatibility (*overdetermined problem*) *above*, we will look for a harmonic function f, regular at infinity that *minimizes the functional*

$$\Phi(f) = \int_{\Omega} (f - T)^2 \, d\mathbf{x}$$

We suppose that $f \in H_2(\Omega_{ext})$, where $H_2(\Omega_{ext})$ is a space of harmonic functions with *inner product*

$$(f, g) \equiv \int_{\Omega_{ext}} \frac{1}{r^2} fg \, d\mathbf{x}$$

The functional Φ attains its minimum in $H_2(\Omega_{ext})$. Hence, assuming Φ has its minimum at a point $f \in H_2(\Omega_{ext})$, its *Gâteaux' differentials equals zero* at f. This yields

$$\int_{\Omega} f v \, d\mathbf{x} = \int_{\Omega} T v \, d\mathbf{x} \qquad \textbf{(18)}$$

for all $v \in H_2(\Omega_{ext})$.

***Eq.* (*18*) represents *Euler's necessary condition* for Φ to have a minimum at f. It is a *starting point for obtaining the function* f .**

We put $v_{nm} = (R_i / r)^{n+1} Y_{nm}(\varphi, \lambda)$, denoting by Y_{nm} *Laplace' surface spherical harmonics.*

Subsequently, $f = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{m=n} f_{nm} v_{nm}$, while f_{nm} are scalar coefficients. After some algebra we easily obtain

$$f = \sum_{n=0}^{\infty} \left(\frac{R_i}{r} \right)^{n+1} \left[T_n^{(i)} + \alpha_n T_n^{(e)} \right] \quad \text{with} \quad \alpha_n = \frac{(2n-1)(1-q^2)}{2(1-q^{2n-1})} q^{n-2}$$

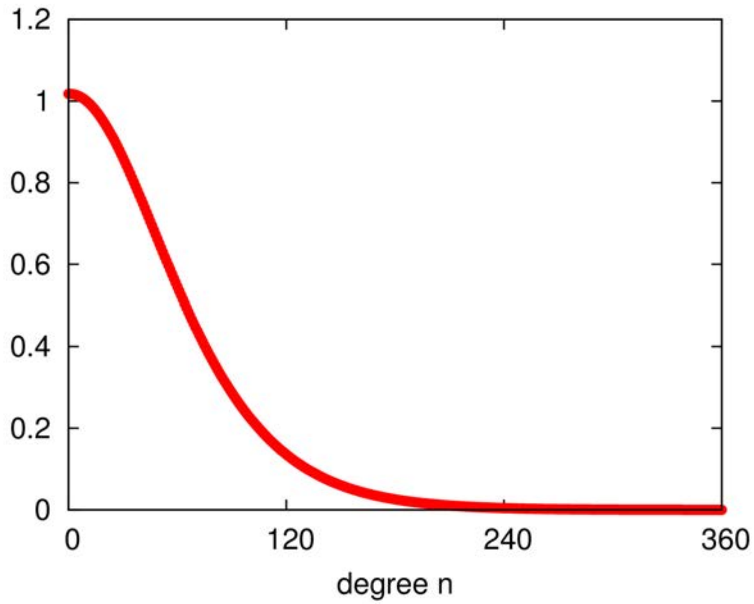


Figure 5. **Values of α_n for**

$$R_i = 6378{,}136\, km$$

and

$$R_e = R_i + 224\, km$$

i.e., for

$$q = 0.966071588$$

The *optimized solution* f is partially generated by $T_n^{(e)}$, but in contrast to the original *Eq.* (*3*), i.e., $T = T^{(i)} + T^{(e)}$ the influence of $T_n^{(e)}$ is now *attenuated* by the factor α_n.

This is illustrated for *GO_CONS_GCF_2_TIM_R5* model in *Fig. 6*.

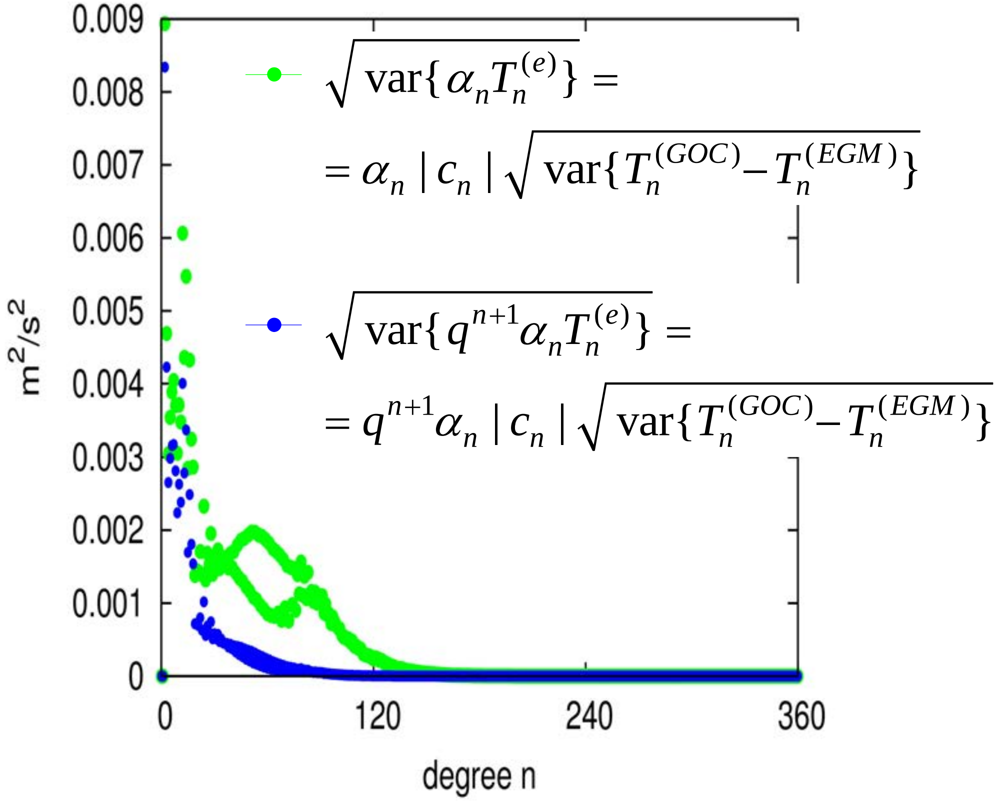


Figure 6

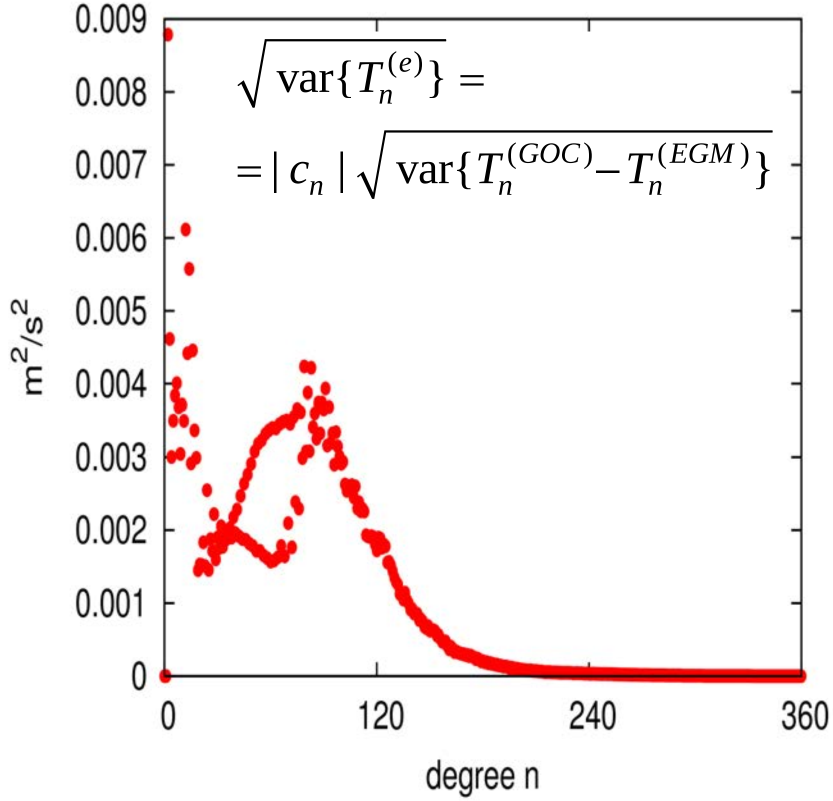


Figure 2 – ***for comparison***

5. Optimized Solution – Influence of Input Data

To see the influence **of the input data δg and t on the optimized solution f** ***we have to return to the original structure*** **of the harmonics $T_n^{(i)}$ and $T_n^{(e)}$.**

Therefore, we insert from *Eqs* (*4a*) and (*4b*) and subsequently obtain

$$f=\sum_{n=0}^{\infty}\left(\frac{R_i}{r}\right)^{n+1}\left[A_n^{(i)}\frac{R_i}{n+1}\delta g_n+A_n^{(e)}t_n\right] \qquad \textbf{(20)}$$

with

$$A_n^{(i)}=\frac{(n+1)(1-\alpha_n q^{n+1})}{D_n} \quad \text{and} \quad A_n^{(e)}=\frac{n\,q^n+\alpha_n(n+1)}{D_n} \qquad \textbf{(21)}$$

where

$$D_n=n(1+q^{2n+1})+1$$

The values of $A_n^{(i)}$ and $A_n^{(e)}$ are in *Fig. 7* that follows.

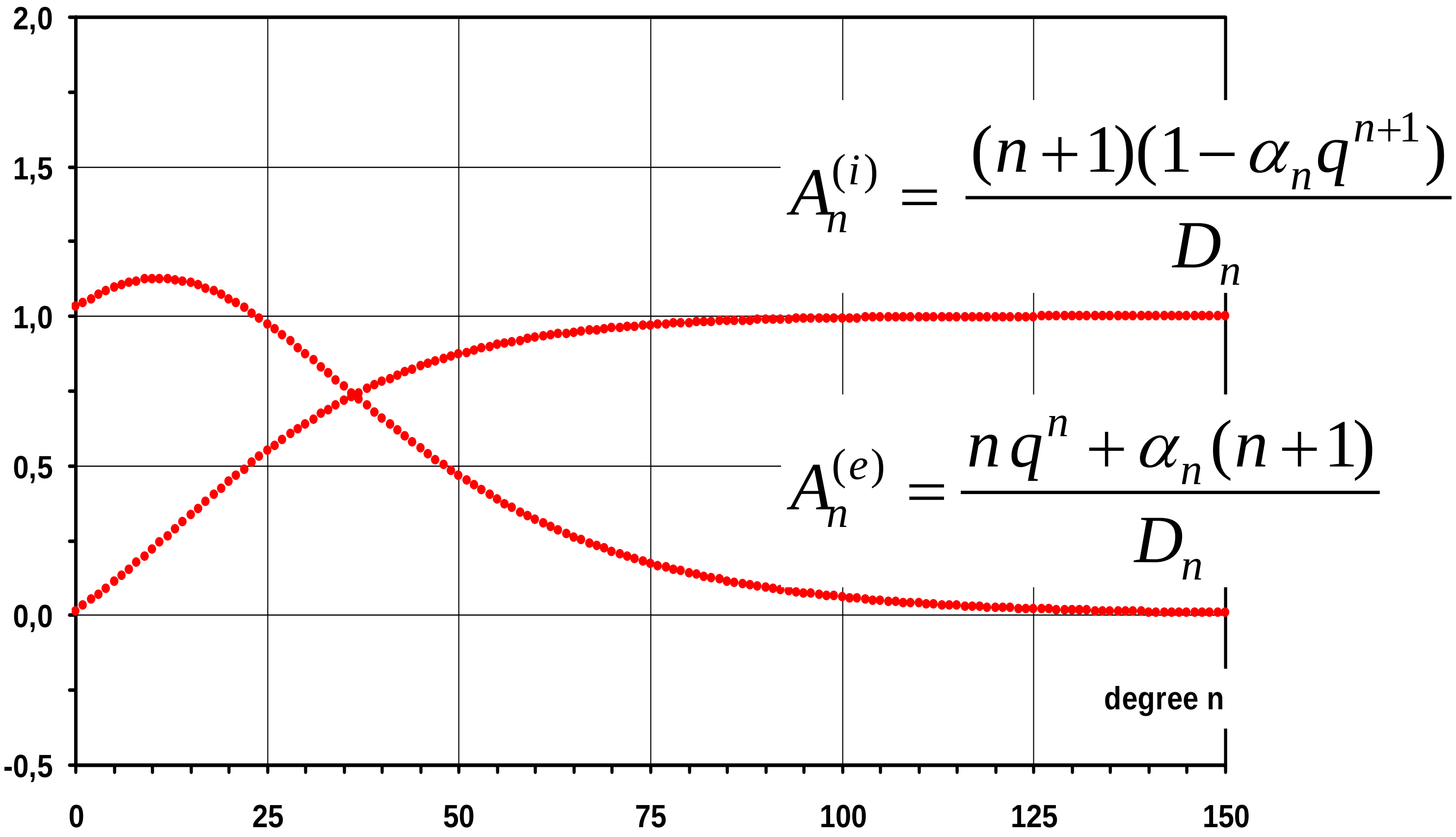


Figure 7. **The coefficients $A_n^{(i)}$ and $A_n^{(e)}$ for $q = 0.966071588$ (i.e. $R_e = R_i + 224\,km$) in case that *gravity disturbances* δg are combined with t representing the *input from a satellite-only model*.**

4. Optimization in $\boldsymbol{H_2^{(1)}}$ - Energetic Concept

Let $H_2^{(1)}(\Omega_{ext})$ be the space of harmonic functions on Ω_{ext} which is equipped with inner product

$$(f,g)_1 = \int_{\Omega_{ext}} \langle \boldsymbol{grad}\ f, \boldsymbol{grad}\ g \rangle \, d\mathbf{x}$$

where $\langle .,. \rangle$ is the scalar product of two vectors in $\boldsymbol{R}^3$. We look for a function $f \in H_2^{(1)}(\Omega_{ext})$ that *minimizes the functional*

$$\Psi(f) = \int_{\Omega} |\,\boldsymbol{grad}\,(f - T)\,|^2 \, d\mathbf{x}$$

Similarly as above the functional Ψ attains its minimum in $H_2^{(1)}(\Omega_{ext})$ and f is defined by the integral identity

$$\int_{\Omega} \langle \boldsymbol{grad}\ f, \boldsymbol{grad}\, v \rangle \, d\mathbf{x} = \int_{\Omega} \langle \boldsymbol{grad}\, T, \boldsymbol{grad}\, v \rangle \, d\mathbf{x}$$

which holds for all $v \in H_2^{(1)}(\Omega_{ext})$.

Interpreting the identity in terms of our function basis, we again write $f=\Sigma_{n=0}^{\infty}\Sigma_{m=-n}^{m=n} f_{nm}v_{nm}$ **, but now we arrive at**

$$f=\sum_{n=0}^{\infty}\left(\frac{R_i}{r}\right)^{n+1}T_n^{(i)}$$

which is considerably more simple. Subsequently we obtain

$$f=\sum_{n=0}^{\infty}\left(\frac{R_i}{r}\right)^{n+1}\left[A_n^{(i)}\frac{R_i}{n+1}\delta g_n+A_n^{(e)}t_n\right]$$

with

$$A_n^{(i)}=\frac{n+1}{D_n} \quad \textbf{and} \quad A_n^{(e)}=\frac{n\,q^n}{D_n}$$

where

$$D_n=n(1+q^{2n+1})+1$$

The values of $A_n^{(i)}$ **and** $A_n^{(e)}$ **are in** ***Fig. 8*** **that follows.**

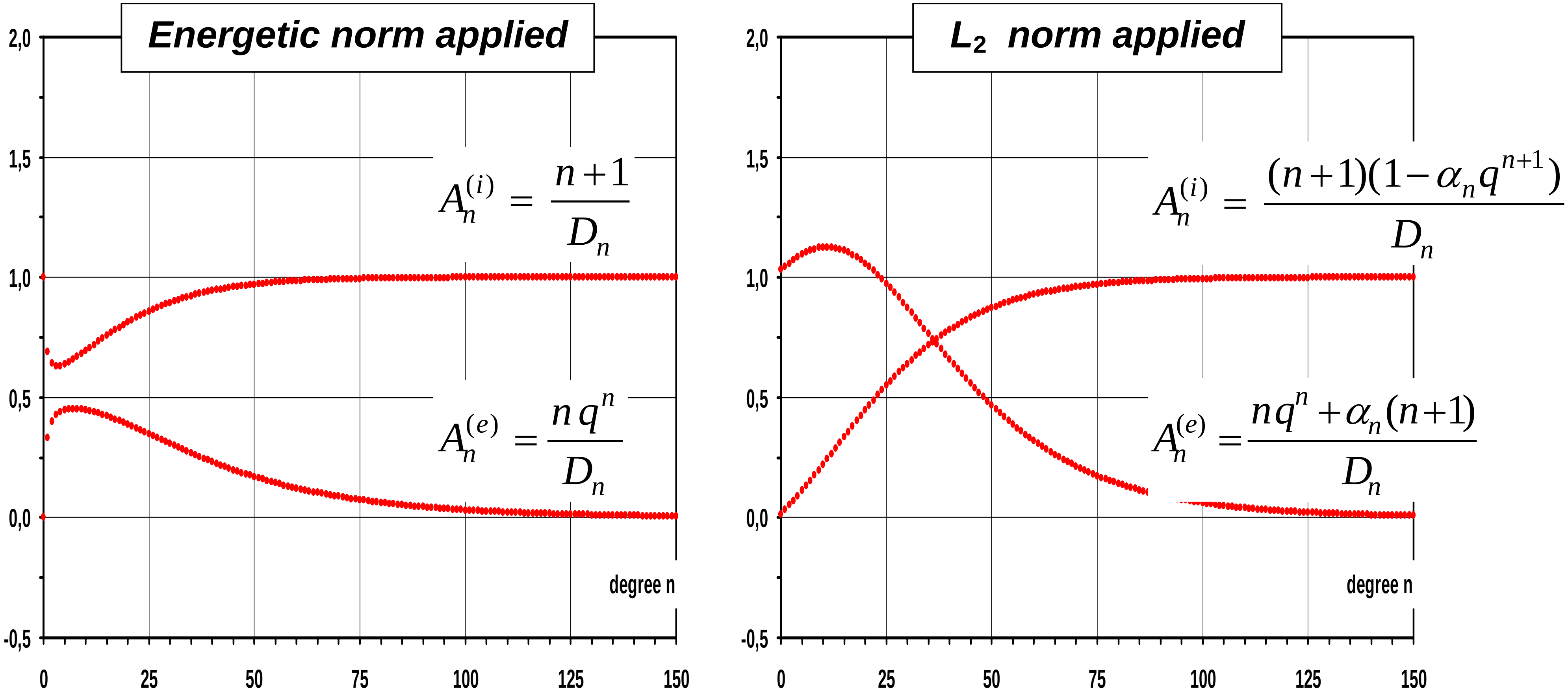


Figure 8. **The coefficients** $A_n^{(i)}$ **and** $A_n^{(e)}$ **for** $R_e = R_i + 224\,km$ **in case that** ***gravity disturbances*** δg **are combined with** t **representing the** ***input from a satellite-only model*** **and (Left) an** ***energetic norm is applied*** **in the optimization concept.**

6. Terrestrial Term and the Integral Kernel

Recall that we obtained the following result - *optimized solution*:

$$f = \sum_{n=0}^{\infty} \left(\frac{R_i}{r} \right)^{n+1} \left[A_n^{(i)} \frac{R_i}{n+1} \delta g_n + A_n^{(e)} t_n \right]$$

***Our aim is* to find an *integral representation* for**

$$f_{terr} = \sum_{n=0}^{\infty} \left(\frac{R_i}{r} \right)^{n+1} A_n^{(i)} \frac{R_i}{n+1} \delta g_n \quad , \qquad A_n^{(i)} = \frac{(n+1)(1-\alpha_n q^{n+1})}{n(1+q^{2n+1})+1}$$

which is the *terrestrial term* in the structure of the optimized solution. We get

$$f_{terr} = \frac{R_i}{4\pi} \int_{\sigma} K^*(r,\psi) \delta g \, d\sigma$$

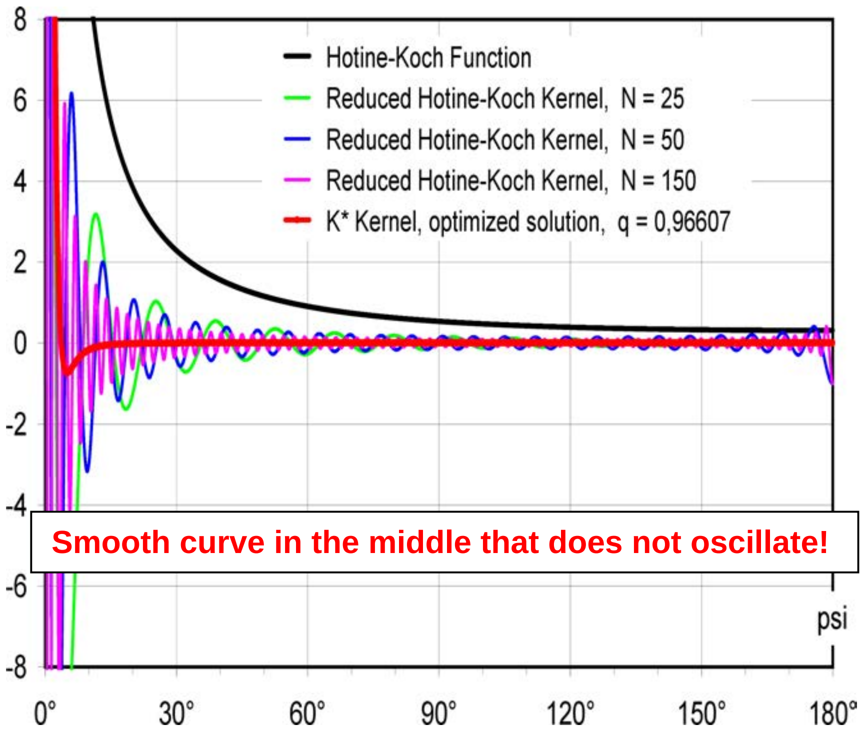
with

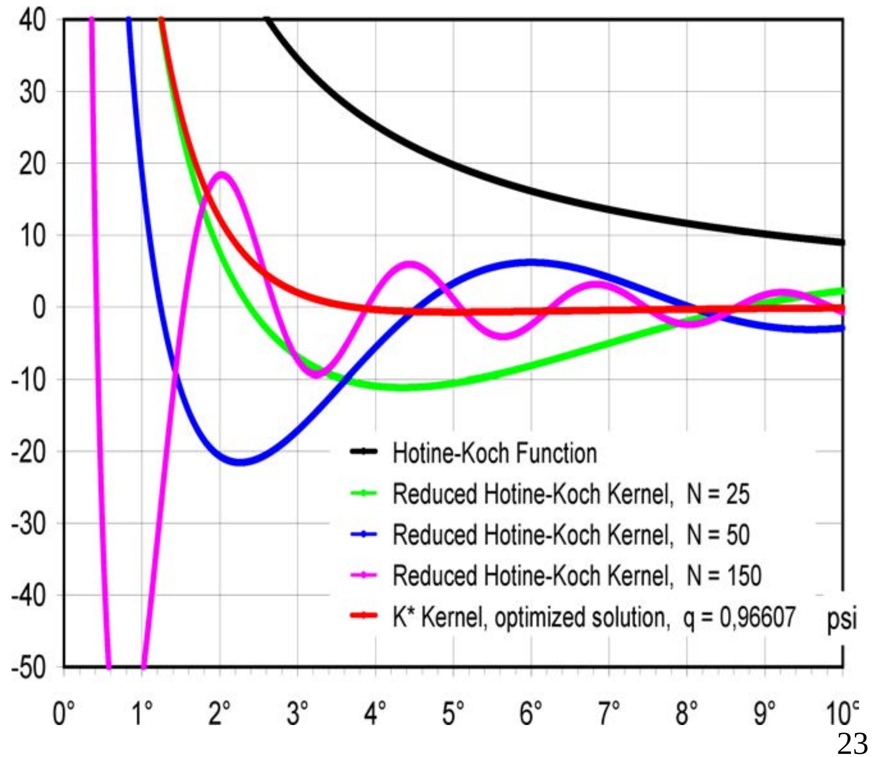
$$K^*(r,\psi) = \sum_{n=0}^{\infty} A_n^{(i)} \frac{2n+1}{n+1} \left(\frac{R_i}{r} \right)^{n+1} P_n(\cos\psi)$$

Of course, we are interested in a closed form of the kernel

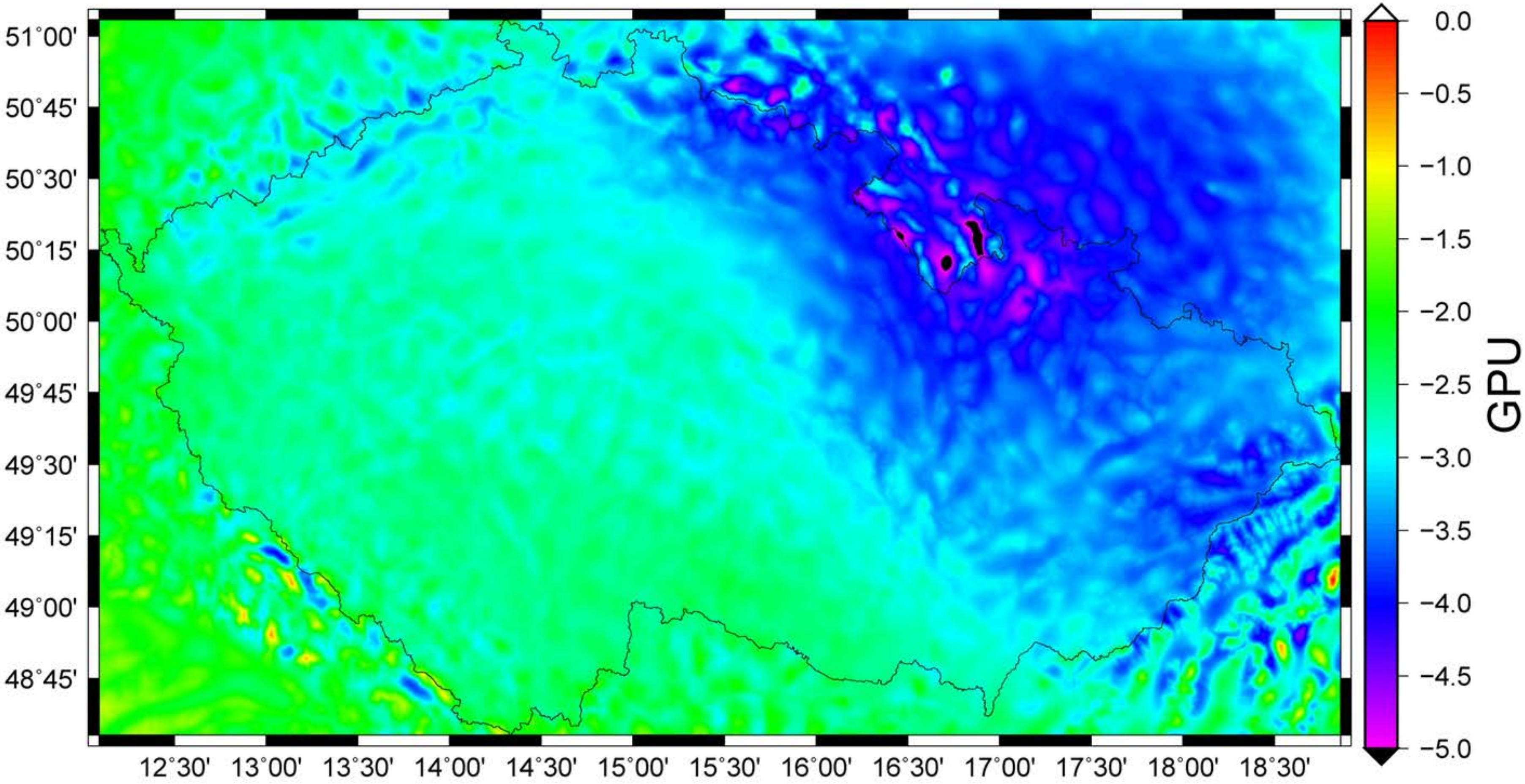
$$K^*(r,\psi) = \sum_{n=0}^{\infty} A_n^{(i)} \frac{2n+1}{n+1} \left(\frac{R_i}{r}\right)^{n+1} P_n(\cos\psi)$$

Here we confine ourselves just to an illustration showing *how the kernel* $K^*(r,\psi)$ *depends on the angle* ψ in case that $r = R_i$. The dependence is shown in the two figures below.





The kernel was used practically. The dominant part of the *terrestrial term* f_{terr} (with respect to EGM2008) was computed for *data from the territory of the Czech Republic* and it is plotted in the following figure.



The composition of

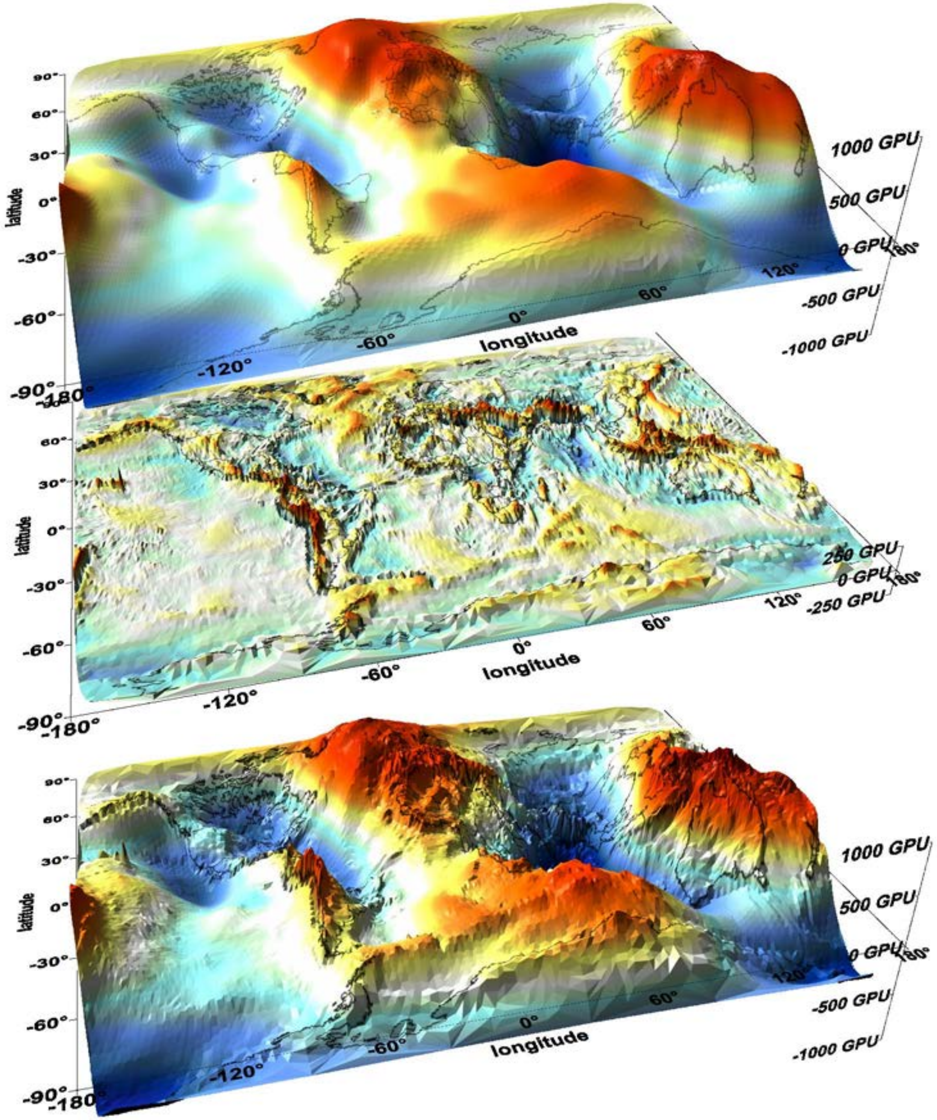
$$f_{sat} = \sum_{n=0}^{\infty} \left(\frac{R_i}{r} \right)^{n+1} A_n^{(e)} t_n$$

and

$$f_{terr} = \sum_{n=0}^{\infty} \left(\frac{R_i}{r} \right)^{n+1} A_n^{(i)} \frac{R_i}{n+1} \delta g_n$$

is then here.

$$f = f_{terr} + f_{sat}$$



Thank you for your attention !

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