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# State Estimation Using the Particle Filter With Mode Tracking

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## 1. Introduction

- Data assimilation is the incorporation of observational data into a numerical model to estimate the model state which accurately describes the observed reality.
- Traditional data assimilation schemes assume that data is modelled by a Gaussian state space model. A particle filter[1] (PF) is an ensemble data assimilation scheme that allows the probability density function (pdf) of non-Gaussian state space models to be approximated.
- Here we consider if the efficiency of the PF in high dimensional space can be improved by the introduction of mode tracking[2].
- When mode tracking we must split the model state into two subspaces. One subspace is forecast using the ordinary PF, the other is estimated using the mode of the marginal pdf. Here we test one hypothesis on how the state should be spilt.
- We compare results from the particle filter and particle filter with mode tracking (PF-MT) to see if the introduction of mode tracking has improved the accuracy of the result.
- Finally we consider the effect model bias has on the solutions from both the PF and PF-MT.

## 2. Particle Filters

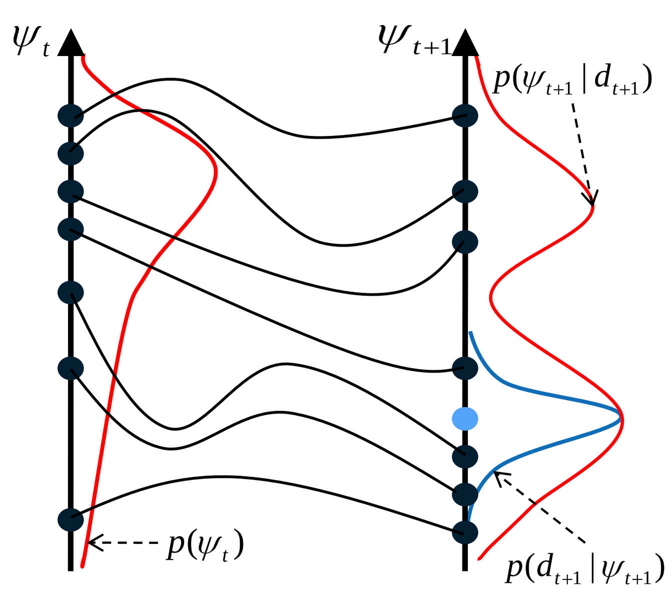


***Figure 1. Schematic diagram of the particle filter. Black dots represent particles. The blue dot is an observation.***

A PF is an ensemble data assimilation scheme that allows non-linear, non-Gaussian state space models to be approximated. A schematic of the PF is given in Figure 1. The PF samples from the probability density of the state to create a number of particles that are then forecast using a numerical model. The information from these particles is combined with observational information and used to estimate the state and its uncertainty at the forecast time.

Unfortunately the number of particles required to accurately forecast the state grows exponentially as the size of the state increases[3]. For this reason Vaswani introduced the idea of mode tracking[2] (MT). It is thought that introducing mode tracking in the particle filter may reduce the number of particles required to achieve a given filter accuracy and therefore increase the efficiency of the PF.

## 3. The Particle Filter with Mode Tracking

When mode tracking, instead of trying to represent the full pdf, we approximate the highest probability state, or mode, of the marginal pdf. To do this the state is split into two subspaces. One subspace is forecast using the PF, the other is treated so that its values are equal to the mode of the marginal pdf. The particle filtering with mode tracking (PF-MT) algorithm is given in Table I.

| The PF-MT Algorithm |
|---|
| Initialization:<br>1. Set $k=0$ and sample $N$ times from the importance function $\pi(\psi_0)$ to give the initial ensemble $\{\psi_0^i\}_{i=1}^N$.<br>2. Set the weights, $\omega_0^i = 1/N$<br><br>For times $k = 1, 2, \ldots$.<br>1. Importance Sample $\Psi_{k,s}$: For $i = 1, 2, \ldots, N$, sample $\Psi_{k,s}^i \sim p(\Psi_{k,s}^i \mid \psi_{k-1}^i)$.<br>2. Mode track $\Psi_{k,r}$: For $i = 1, 2, \ldots, N$, set $\Psi_{k,r} = m_k^i$ where $m_k^i(\psi_{k-1}^i, \Psi_{k,s}^i, d_k) = \arg\min_{\Psi_{k,r}}[-\log p(d_k \mid \Psi_{k,s}^i) p(\Psi_{k,r} \mid \psi_{k-1}^i, \Psi_{k,s}^i)]$.<br><br>3. Weight: For $i = 1, 2, \ldots N$, compute $w_k^i = \frac{\omega_k^i}{\sum_{j=1}^N \omega_k^j}$ where $\psi_k^i = [\Psi_{k,s}^i, \Psi_{k,r}^i]$.<br>and $\omega_k^i = w_{k-1}^i p(d_k \mid \psi_k^i) p(\Psi_{k,r}^i \mid \psi_{k-1}^i, \Psi_{k,s}^i)$.<br><br>4. Resample: Replicate particles in proportion to their weights and reset the weights $\omega_k^i = 1/N$. |

***Table I. The PF – MT algorithm***

## 4. The Numerical Model

To test the behaviour of the PF-MT we use a discretisation of the stochastic Lorenz equations,

$$dX = \sigma(Y - X)dt + B_X dW_X,$$
$$dY = (X(\rho - Z) - Y)dt + B_Y dW_Y,$$
$$dZ = (XY - \beta Z)dt + B_Z dW_Z,$$

where *X*, *Y* and *Z* represent random variables and, σ = 10, ρ = 28 and β = ⁸⁄₃. For the unbiased model $dW_X$, $dW_Y$ and $dW_Z$ represent independent Wiener processes. The constants $B_X$, $B_Y$, $B_Z$ are parameters governing the magnitude of the noise in the equation. For simplicity we set each to be the same value *B = 0.1*. To form a biased model a value is assigned to each of $dW_X$, $dW_Y$ and $dW_Z$, this value is kept constant at each timestep.

## 5. Which States Should we Mode Track?

When using the PF-MT it is important that the correct states are mode tracked so that the most accurate result is obtained. Vaswani suggests that the best results should be obtained when we mode track the maximum number of unimodal dimensions. In our case the conditional pdf is always unimodal so we expect to see the best results when we mode track two dimensions. We test this hypothesis by comparing the root mean squared errors (RMSEs) of the mean of the ensemble obtained from the PF-MT after splitting the state in different ways. The results are summarised in the table II. From Table II it can be seen that the best results are not always obtained when we mode track two dimensions, perhaps due to the complicated nonlinear structure of the full problem.

| Number of Particles | Best States to Mode Track | Worst States to Mode Track |
|---|---|---|
| 5 | Z | Y |
| 50 | Z | X,Y |
| 500 | X,Z | X,Y |

***Table II. Best and worst states to mode track when considering the RMSE***

## 6. Comparison of the PF and PF-MT

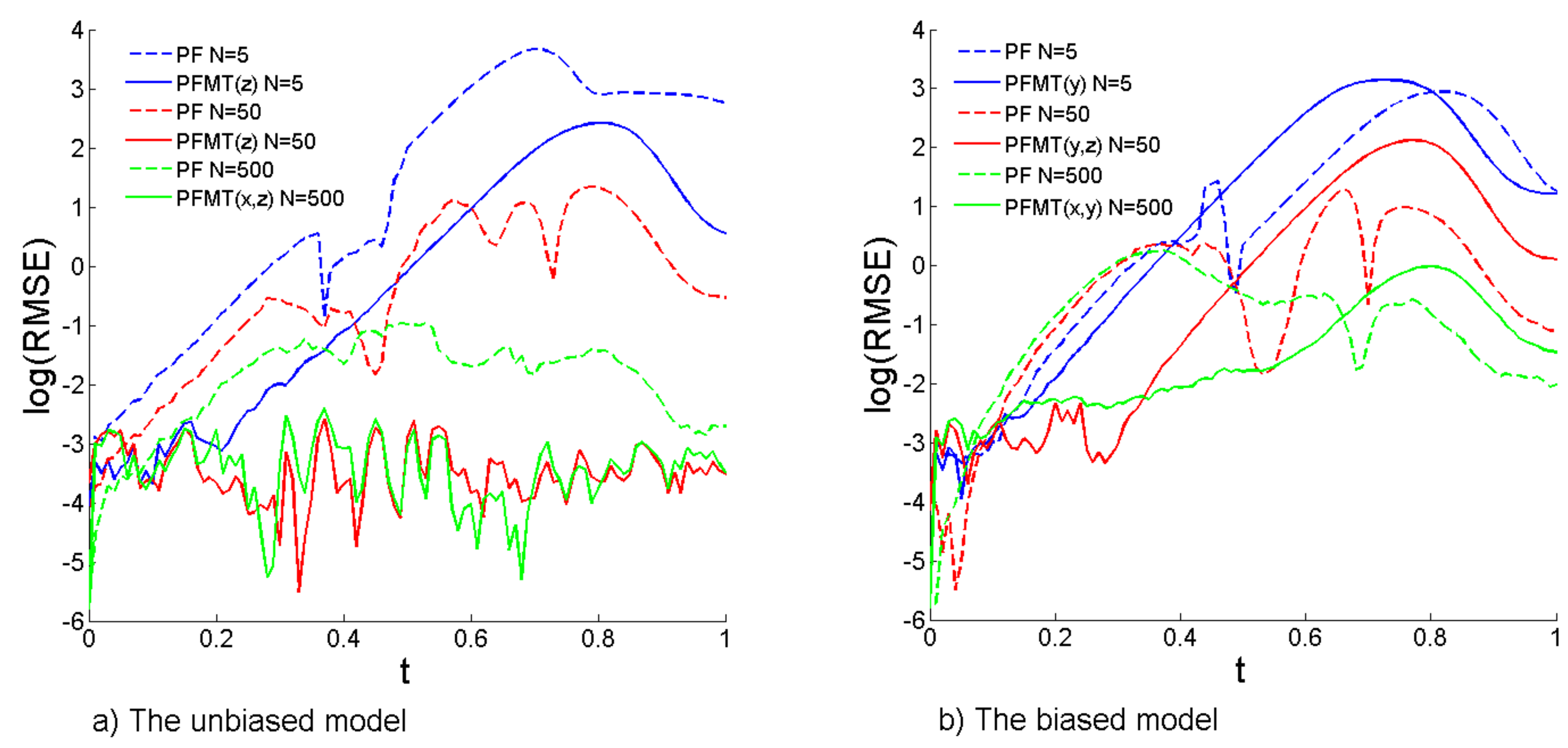


***Figure 2. RMSE of the mean of the ensemble for the PF and PF-MT using different numbers of particles***

Figures 2a and 2b show the RMSE for results from the PF and PF-MT using both a biased an unbiased model. For both the PF and PF-MT it is found that increasing the number of particles increases the accuracy of the solution obtained. When using the unbiased model it is found that the introduction of mode tracking reduces the RMSE for a given number particles. This suggests that in general the PF-MT can produce the same accuracy as the PF but with fewer particles. Therefore the introduction of mode tracking can make the particle filter more efficient in high dimensional space. When the model is biased it is found that for small ensemble sizes the PF can provide more accurate results than the PF-MT. Hence for the biased model it is possible that the introduction of mode tracking will not improve the efficiency of the PF.

## 7. Conclusions

- We tested the hypothesis that the best results should be obtained when two states were mode tracked. For small numbers of particles it was not the case, possibly due to the complicated nonlinear structure of the full problem.
- When using an unbiased model the introduction of mode tracking reduced the RMSE for a given number of particles. This suggests that the introduction of mode tracking can improve the efficiency of the PF in high dimensions.
- When a biased model is used it is found that for small ensemble sizes the PF can provide more accurate results than the PF-MT.

**References**

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