Basics about radiative transfer

Bruno Carli

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Radiative Transfer Equation

The specific intensity of radiation is the energy flux per unit time, unit frequency, unit solid angle and unit area normal to the direction of propagation.

The radiative transfer equation states that the specific intensity of radiation I_σ during its propagation in a medium is subject to losses due to extinction and to gains due to emission:

$$\frac{dI_\sigma}{dx} = -\mu_\sigma \cdot I_\sigma + \rho \cdot j_\sigma$$

where x is the co-ordinate along the optical path, μ_σ is the extinction coefficient, ρ is the mass density j_σ is the emission coefficient per unit mass.

A Simple Case

As a start it is useful to study the radiative transfer equation in the simple case of:

- no scattering effect,
- local thermodynamic equilibrium,
- homogeneous medium.

Extinction

In general, the extinction coefficient μ_σ includes both the absorption coefficient α_σ and the scattering coefficient s_σ, of both the gas and the aerosols present in the gas:

$$\mu_\sigma = \alpha_\sigma{}^{gas} + s_\sigma{}^{gas} + \alpha_\sigma{}^{aerosol} + s_\sigma{}^{aerosol}$$

In the case of a pure gas atmosphere with no-scattering a simple expression is obtained:

$$\mu_\sigma = \alpha_\sigma{}^{gas} = \alpha_\sigma$$

Emission

In absence of scattering and for local thermodynamic equilibrium (LTE), the source function is equal to :

$$\rho \cdot j_\sigma = \alpha_\sigma B_\sigma(T)$$

where α_σ is the absorption coefficient (equal to the emission coefficient for the Kirchhoff's law) and $B_\sigma(T)$ is the Plank function at frequency σ and temperature T.

Radiative Transfer Equation

for LTE and No Scattering

For an atmosphere with no scattering and in LTE the radiative transfer equation is reduced to:

$$\frac{dI_\sigma}{dx} = -\alpha_\sigma \cdot I_\sigma + \alpha_\sigma \cdot B_\sigma(T)$$

Note on Conservation of Energy

$$\frac{dI_\sigma}{dx} = -\alpha_\sigma \cdot I_\sigma + \alpha_\sigma \cdot B_\sigma(T)$$

Losses and gains must obey the second law of thermodynamics. For any term that introduces a loss there must be a term that introduces a gain.

In the propagating beam a change of intensity is caused by the difference between the intensity of the source I_σ that is being attenuated and the intensity of the local source $B_\sigma(T)$.

Analytical Solution of the Integral

Homogeneous Medium

An analytical integral expression of the differential equation of radiative transfer:

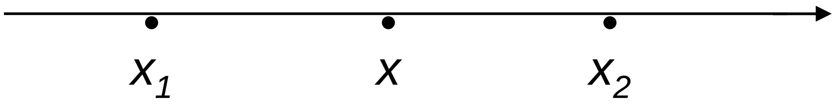
$$\frac{dI_\sigma}{dx} = -\alpha_\sigma \cdot I_\sigma + \alpha_\sigma \cdot B_\sigma(T)$$

can only be obtained for an homogeneous medium.

Analytical Solution of the Integral

Homogeneous Medium

The differential equation is at point x and we want to obtain the integral from x_1 and x_2 .



This can be formally obtained multiplying both terms of the differential equation by $exp[\alpha_\sigma(x-x_1)]$ (i.e. the attenuation from x_1 to x).

$$e^{\alpha_\sigma(x-x_1)}\cdot\frac{dI_\sigma}{dx}=\alpha_\sigma\cdot e^{\alpha_\sigma(x-x_1)}\cdot[-I_\sigma+\cdot B_\sigma(T)]$$

$$e^{\alpha_\sigma(x-x_1)}\cdot\frac{dI_\sigma}{dx}+\alpha_\sigma\cdot e^{\alpha_\sigma(x-x_1)}\cdot I_\sigma=\alpha_\sigma\cdot e^{\alpha_\sigma(x-x_1)}\cdot B_\sigma(T)$$

$$\frac{d}{dx}\left[e^{\alpha_\sigma(x-x_1)}\cdot I_\sigma\right]=\frac{d}{dx}\left[e^{\alpha_\sigma(x-x_1)}\right]\cdot B_\sigma(T)$$

An expression is obtained that can be integrated from x_1 to x_2.

$$e^{\alpha_\sigma(x_2-x_1)}\cdot I_\sigma(x_2)-e^{\alpha_\sigma(x_1-x_1)}\cdot I_\sigma(x_1)=\left(e^{\alpha_\sigma(x_2-x_1)}-e^{\alpha_\sigma(x_1-x_1)}\right)\cdot B_\sigma(T)$$

$$e^{\alpha_\sigma(x_2-x_1)}\cdot I_\sigma(x_2)=I_\sigma(x_1)+B_\sigma(T)\cdot\left(e^{\alpha_\sigma(x_2-x_1)}-1\right)$$

$$I_\sigma(x_2)=I_\sigma(x_1)\cdot e^{-\alpha_\sigma(x_2-x_1)}+B_\sigma(T)\cdot\left(1-e^{-\alpha_\sigma(x_2-x_1)}\right)$$

Analytical Solution of the Integral

Homogeneous Medium

In the integral expression of radiative transfer:

$$I_\sigma(x_2) = I_\sigma(x_1)\cdot e^{-\alpha_\sigma(x_2-x_1)} + B_\sigma(T)\cdot\left(1-e^{-\alpha_\sigma(x_2-x_1)}\right)$$

the first term is the Lambert-Beer law which gives the attenuation of the external source and the second term gives the emission of the local source.

The Complications

The modelling of radiative transfer is made more complicated by :

- scattering,
- non-LTE,
- variable medium.

These will be individually considered, the simultaneous application of more than one complication is only an analytical problem.

Scattering

In presence of scattering:

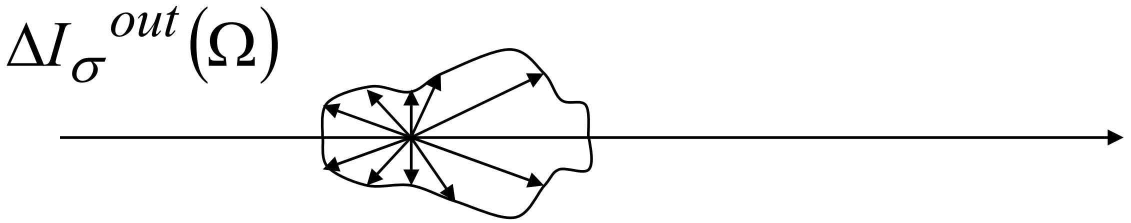
$$\mu_\sigma = \alpha_\sigma{}^{gas} + s_\sigma{}^{gas} + \alpha_\sigma{}^{aerosol} + s_\sigma{}^{aerosol} = \alpha_\sigma + s_\sigma$$

the differential equation is equal to:

$$\frac{dI_\sigma}{dx} = -\alpha_\sigma \cdot I_\sigma \underbrace{- s_\sigma \cdot I_\sigma}_{\text{loss}} + \alpha_\sigma \cdot B_\sigma(T) + \underbrace{s_\sigma \cdot J_\sigma}_{\text{gain}}$$

Scattering

Loss



For each path *Δ x,* the amplitude of the scattered intensity $\Delta I_\sigma^{out}(\Omega)$ in each direction Ω is measured by the scattering phase function $p_\sigma(\Omega)$:

$$\Delta I_\sigma^{out}(\Omega) = s_\sigma \cdot p_\sigma(\Omega) \cdot I_\sigma \cdot \Delta x$$

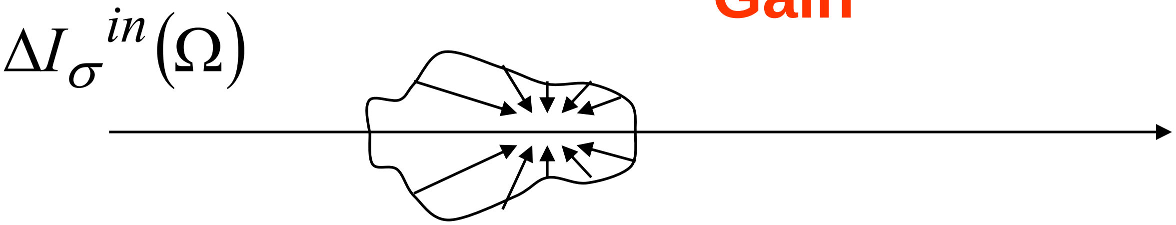
with:

$$\int p_\sigma(\Omega) \cdot d\Omega = 1$$

$$\int \Delta I_\sigma^{out}(\Omega) \cdot d\Omega = \int s_\sigma \cdot p_\sigma(\Omega) \cdot I_\sigma \cdot d\Omega \cdot \Delta x = s_\sigma \cdot I_\sigma \cdot \Delta x$$

Scattering

Gain



The amplitude of the intensity $\Delta I_\sigma{}^{in}(\Omega)$ scattered into the beam from each direction Ω is measured by the scattering phase function $p_\sigma(\Omega)$:

$$\Delta I_\sigma{}^{in}(\Omega) = s_\sigma \cdot p_\sigma(\Omega) \cdot I_\sigma(\Omega) \cdot \Delta x$$

The total contribution is equal to:

$$\int \Delta I_\sigma{}^{in}(\Omega) \cdot d\Omega = \int s_\sigma \cdot p_\sigma(\Omega) \cdot I_\sigma(\Omega) \cdot d\Omega \cdot \Delta x = s_\sigma \cdot J_\sigma \cdot \Delta x$$

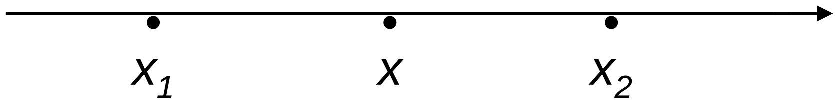
where the source function J_σ is defined as:

$$J_\sigma \overset{\text{def}}{=} \int p_\sigma(\Omega) \cdot I_\sigma(\Omega) \cdot d\Omega$$

Analytical Solution of the Integral Equation

Homogeneous Scattering Medium

The differential equation is at point x and we want to obtain the integral from x_1 and x_2 .



This can be formally obtained multiplying both terms of the differential equation by $exp[\alpha_\sigma+s(x-x_1)]$ (i.e. the attenuation from x_1 to x).

$$e^{(\alpha_\sigma+s)(x-x_1)}\cdot\frac{dI_\sigma}{dx}=-(\alpha_\sigma+s)\cdot e^{(\alpha_\sigma+s)(x-x_1)}\cdot I_\sigma+\alpha_\sigma\cdot e^{(\alpha_\sigma+s)(x-x_1)}\cdot B_\sigma(T)+s_\sigma\cdot e^{(\alpha_\sigma+s)(x-x_1)}\cdot J_\sigma$$

$$e^{(\alpha_\sigma+s)(x-x_1)}\cdot\frac{dI_\sigma}{dx}+(\alpha_\sigma+s)\cdot e^{(\alpha_\sigma+s)(x-x_1)}\cdot I_\sigma=\alpha_\sigma\cdot e^{(\alpha_\sigma+s)(x-x_1)}\cdot B_\sigma(T)+s\cdot e^{(\alpha_\sigma+s)(x-x_1)}\cdot J_\sigma$$

$$\frac{d}{dx}[e^{(\alpha_\sigma+s)(x-x_1)}\cdot I_\sigma]=\frac{\alpha_\sigma}{\alpha_\sigma+s}\frac{d}{dx}[e^{(\alpha_\sigma+s)(x-x_1)}]\cdot B_\sigma(T)+\frac{s}{\alpha_\sigma+s}\frac{d}{dx}[e^{(\alpha_\sigma+s)(x-x_1)}]\cdot J_\sigma$$

An expression is obtained that can be integrated from x_1 to x_2.

$$e^{(\alpha_\sigma+s)(x_2-x_1)}\cdot I_\sigma(x_2)-I_\sigma(x_1)=\frac{\alpha_\sigma}{(\alpha_\sigma+s)}\cdot\left[e^{(\alpha_\sigma+s)(x_2-x_1)}-1\right]\cdot B_\sigma(T)+\frac{s}{(\alpha_\sigma+s)}\cdot\left[e^{(\alpha_\sigma+s)(x_2-x_1)}-1\right]\cdot J_\sigma$$

$$e^{(\alpha_\sigma+s)(x_2-x_1)}\cdot I_\sigma(x_2)=I_\sigma(x_1)+\frac{\alpha_\sigma}{(\alpha_\sigma+s)}\cdot\left[e^{(\alpha_\sigma+s)(x_2-x_1)}-1\right]\cdot B_\sigma(T)+\frac{s}{(\alpha_\sigma+s)}\cdot\left[e^{(\alpha_\sigma+s)(x_2-x_1)}-1\right]\cdot J_\sigma$$

$$I_\sigma(x_2)=I_\sigma(x_1)\cdot e^{-(\alpha_\sigma+s)(x_2-x_1)}+\frac{\alpha_\sigma}{(\alpha_\sigma+s)}\cdot\left[1-e^{-(\alpha_\sigma+s)(x_2-x_1)}\right]\cdot B_\sigma(T)+\frac{s}{(\alpha_\sigma+s)}\cdot\left[1-e^{-(\alpha_\sigma+s)(x_2-x_1)}\right]\cdot J_\sigma$$

Scattering

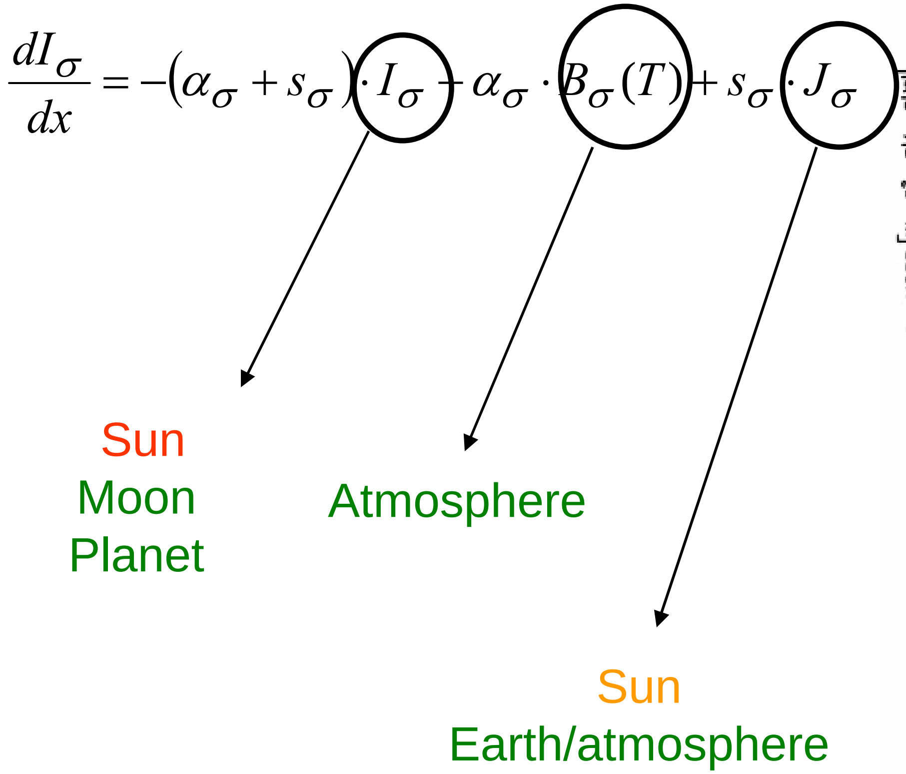
Therefore, in presence of scattering the differential equation of radiative transfer is :

$$\frac{dI_\sigma}{dx} = -\left(\alpha_\sigma + s_\sigma\right)\cdot I_\sigma + \alpha_\sigma \cdot B_\sigma(T) + s_\sigma \cdot J_\sigma$$

and the solution over an homogeneous path from x_1 to x_2 is equal to:

$$I_\sigma\left(x_2\right) = I_\sigma\left(x_1\right)\cdot e^{-(\alpha_\sigma + s_\sigma)\cdot(x_2 - x_1)} + \frac{\alpha_\sigma \cdot B_\sigma(T) + s_\sigma \cdot J_\sigma}{(\alpha_\sigma + s_\sigma)}\cdot\left(1 - e^{-(\alpha_\sigma + s_\sigma)\cdot(x_2 - x_1)}\right)$$

Relative contributions

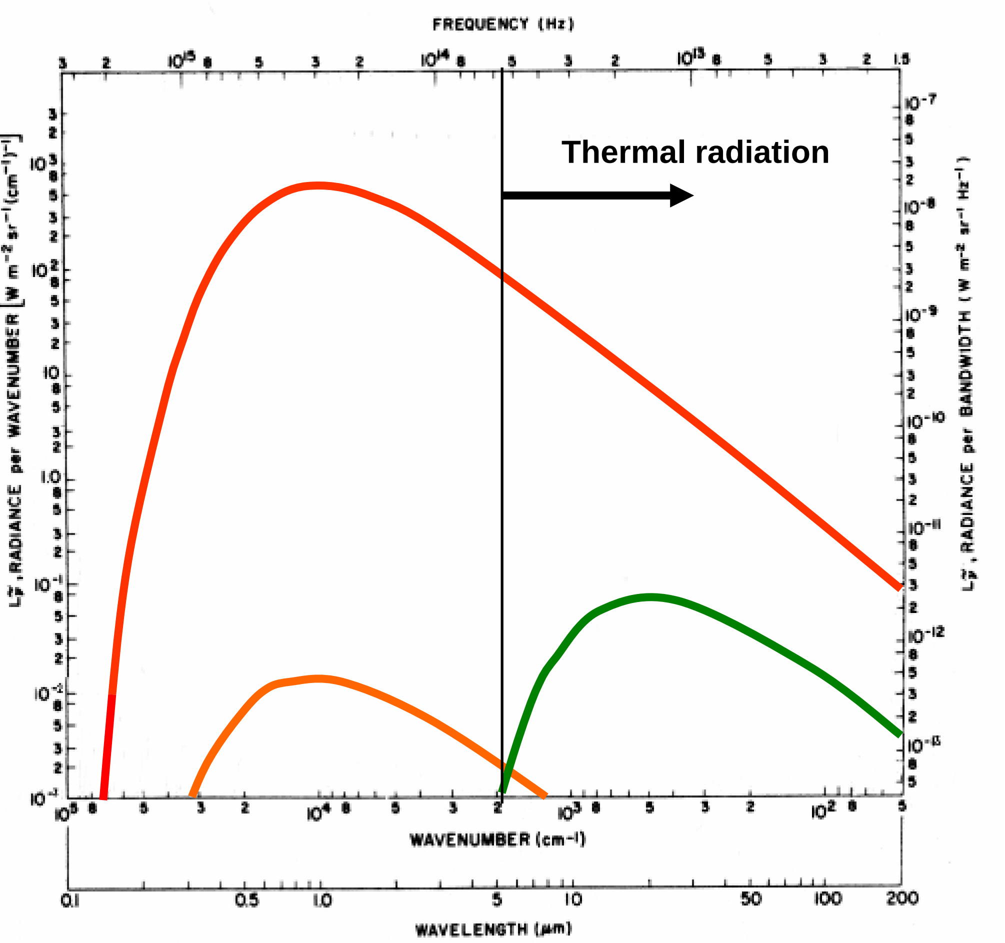


$$\frac{dI_\sigma}{dx} = -(\alpha_\sigma + s_\sigma)\cdot \boxed{I_\sigma} + \alpha_\sigma \cdot \boxed{B_\sigma(T)} + s_\sigma \cdot \boxed{J_\sigma}$$

Sun
Moon
Planet

Atmosphere

Sun
Earth/atmosphere



LTE

Radiative transfer is an exchange of energy between the radiation field and the energy levels of molecules and atoms (which are defined by the Boltzman temperature).

We are in local thermodynamic equilibrium (LTE) when the Boltzman temperature is in equilibrium with the kinetic temperature.

Of course LTE does not imply a complete equilibrium that includes the radiation field. When an equilibrium exists between the radiation field and the local black-body emission no energy exchange and no radiative transfer occur.

Non-LTE

The Boltzman temperature is controlled by chemical reaction, radiation absorption and thermal collisions.
When the collisions are not frequent enough the Boltzman temperature can be different from the kinetic temperature and we are in non-LTE conditions.

When in non-LTE conditions we must consider the different components of the medium and define for each of them their individual temperature $T^{(i)}$ and absorption coefficient $\alpha_\sigma^{(i)}$.

Non-LTE

In the case of non-LTE conditions, the differential equation of radiative transfer equation is :

$$\frac{dI_\sigma}{dx} = -\sum_i \alpha_\sigma{}^{(i)} \cdot I_\sigma + \sum_i \alpha_\sigma{}^{(i)} \cdot B_\sigma(T^{(i)})$$

and the solution over a path from x_1 to x_2 is equal to:

$$I_\sigma(x_2) = I_\sigma(x_1) \cdot e^{-\sum_i \alpha_\sigma{}^{(i)}(x_2 - x_1)} + \frac{\sum_i \alpha_\sigma{}^{(i)} B_\sigma(T^{(i)})}{\sum_i \alpha_\sigma{}^{(i)}} \cdot \left(1 - e^{-\sum_i \alpha_\sigma{}^{(i)}(x_2 - x_1)}\right)$$

Non-homogeneous Medium

When the optical and physical properties of the medium are not constant along the optical path, the absorption coefficient $\alpha_\sigma(x)$ and the local temperature $T(x)$ depend on the variable of integration x. In general, for a non-homogeneous medium the differential equation cannot be analytically integrated.

Integral equation of Radiative Transfer

non-homogeneous medium

Intensity of the background source

“Transmittance” between *0* and *L*

“Transmittance” between *l* and *L*

$$I_\sigma(L) = \underbrace{I_\sigma(0)\, e^{-\tau_\sigma(0,L)}}_{\text{Absorption term}} + \int_0^{\tau_\sigma(0,L)} \underbrace{B_\sigma(T(x))\, e^{-\tau_\sigma(x,L)}}_{\text{Emission term}}\, d\tau_\sigma$$

Spectral intensity observed at L

“Optical depth” →

$$\tau_\sigma(x,L) = \int_x^L \alpha_\sigma(x')\,dx'$$

Discrete calculation of Radiative Transfer integral

This integral is numerically performed as:

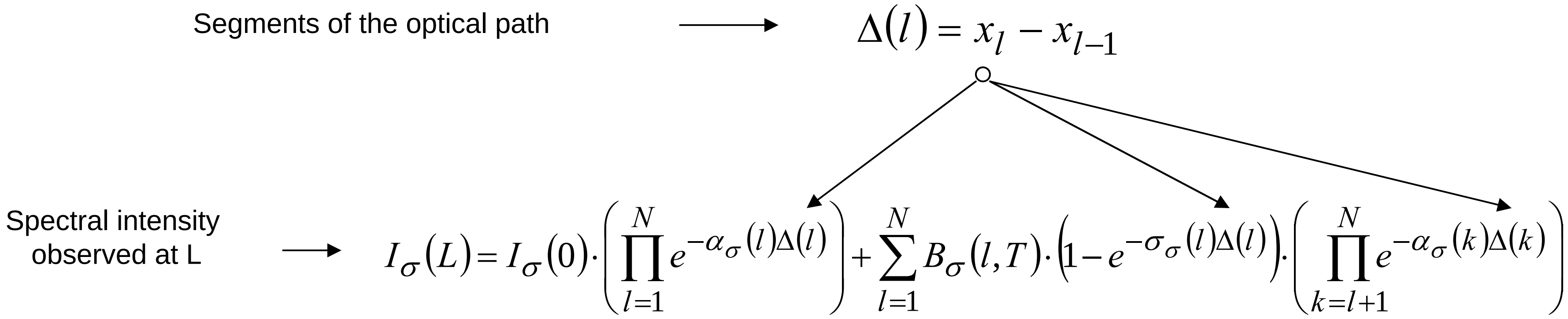
$$I_\sigma(L) = I_\sigma(0) \cdot \left(\prod_{l=1}^{N} e^{-\alpha_\sigma(l)\Delta(l)} \right) +$$

$$+ \sum_{l=1}^{N} B_\sigma(l,T) \cdot \left(1 - e^{-\sigma_\sigma(l)\Delta(l)}\right) \cdot \left(\prod_{k=l+1}^{N} e^{-\alpha_\sigma(k)\Delta(k)} \right)$$

Tree of operations for discrete calculation

segmentation

The optical path is divided in a set of contiguous segments in which the path is straight and the atmosphere has constant properties (segmentation).



Segments of the optical path ⟶

$$\Delta(l) = x_l - x_{l-1}$$

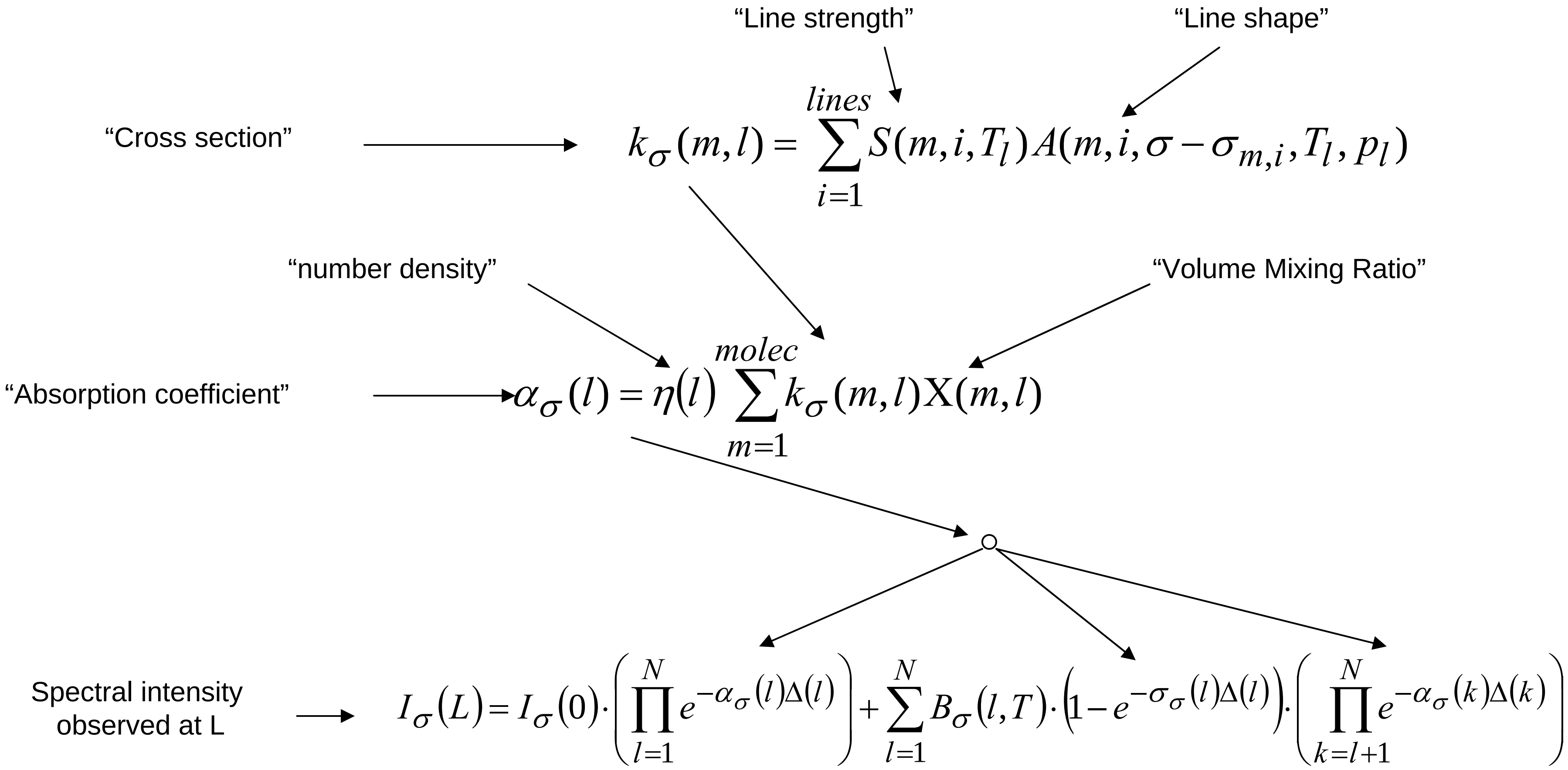
Spectral intensity observed at L ⟶

$$I_\sigma(L) = I_\sigma(0)\cdot\left(\prod_{l=1}^{N} e^{-\alpha_\sigma(l)\Delta(l)}\right) + \sum_{l=1}^{N} B_\sigma(l,T)\cdot\left(1 - e^{-\sigma_\sigma(l)\Delta(l)}\right)\cdot\left(\prod_{k=l+1}^{N} e^{-\alpha_\sigma(k)\Delta(k)}\right)$$

In each segment the absorption coefficient is also calculated.

Tree of operations for discrete calculation

absorption coefficient



"Line strength"

"Line shape"

"Cross section"

$$k_\sigma(m,l) = \sum_{i=1}^{lines} S(m,i,T_l)\,A(m,i,\sigma-\sigma_{m,i},T_l,p_l)$$

"number density"

"Volume Mixing Ratio"

"Absorption coefficient"

$$\alpha_\sigma(l) = \eta(l) \sum_{m=1}^{molec} k_\sigma(m,l)\,\mathrm{X}(m,l)$$

Spectral intensity
observed at L

$$I_\sigma(L) = I_\sigma(0)\cdot\left(\prod_{l=1}^{N} e^{-\alpha_\sigma(l)\Delta(l)}\right) + \sum_{l=1}^{N} B_\sigma(l,T)\cdot\left(1-e^{-\sigma_\sigma(l)\Delta(l)}\right)\cdot\left(\prod_{k=l+1}^{N} e^{-\alpha_\sigma(k)\Delta(k)}\right)$$

Optical Path

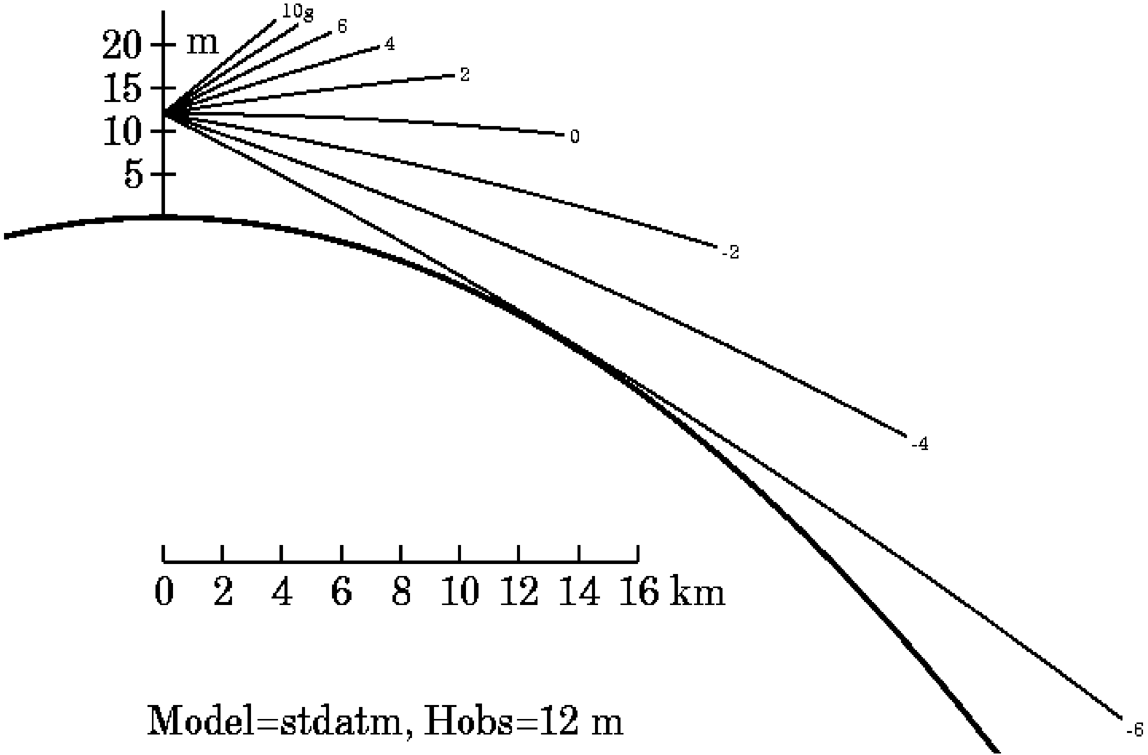
In general, radiative transfer occurs along a straight line and preserves images. However, radiation is subject to refraction and the refractive index of a gas is different from that of vacuum.
The refractive index depends on the composition of the gas, but is in general proportional to the gas density.

When the beam crosses a gradient of density it bends towards the higher density.

Optical Path

Limb View

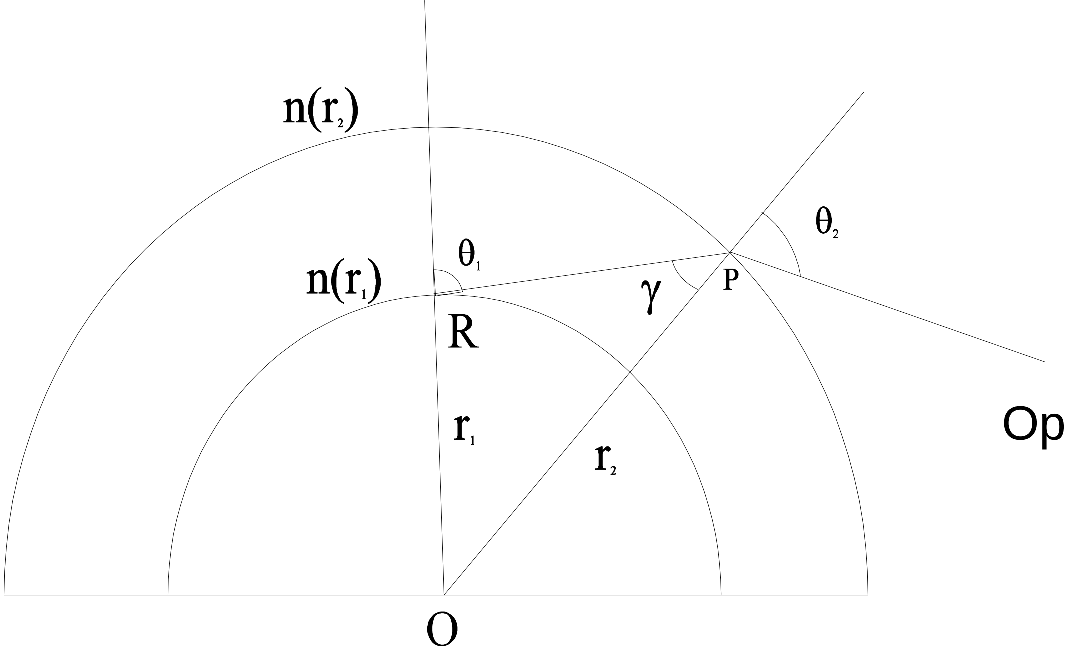
In an atmosphere in hydrostatic equilibrium, the air density increases with decreasing altitude. A line of sight close to the limb view is subject to a curvature that is concave towards the Earth.



Optical path

- Use of Snell's law for optical ray tracing in the atmosphere

$$n(r_2)\cdot\sin\theta_2 = n(r_1)\cdot\sin\gamma$$



Invariant for spherical geometry:

$$r_i\cdot\sin\theta_i = const$$

Optical invariant for spherical geometry:

$$r_i\cdot n(r_i)\cdot\sin\theta_i = const$$

Optical Path

Mirages

Mirages are multiple images formed by atmospheric refraction.
A mirage can only occur below the astronomical horizon.

Several types of mirages are possible, the main ones are:

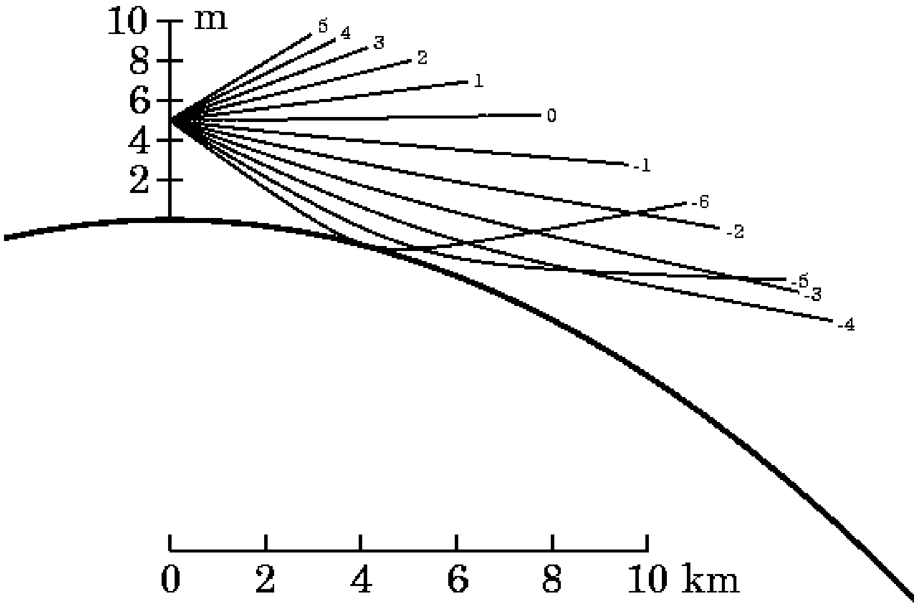
- the inferior mirage, when a reflection-like image appears below the “normal” image

- the superior mirage, when a reflection-like image appears above the “normal” image.

Optical Path

Inferior Mirage

The inferior mirage occurs when the surface of the Earth, heated by the Sun, produces a layer of hot air of lower density just at the surface. Grazing rays bend back up into the denser air above:



Model=inferior mirage, Hobs=5 m

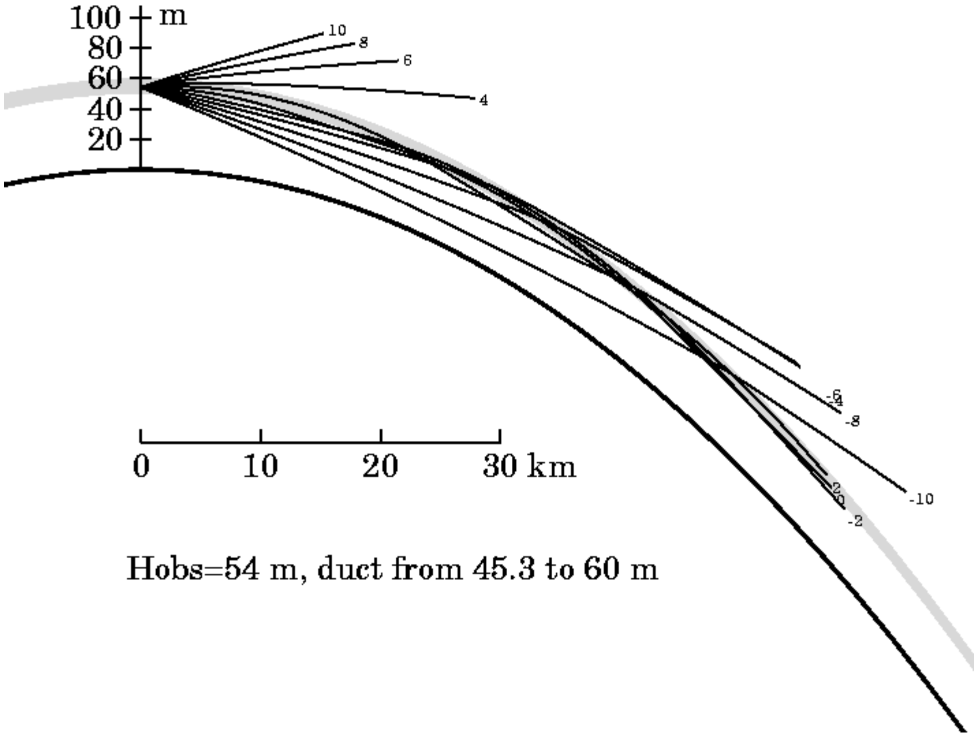
Optical Path

Inferior Mirage

Optical Path

Superior Mirage

The superior mirage requires a more complex atmospheric structure with a cold and high density layer at some altitude above the surface.



Hobs=54 m, duct from 45.3 to 60 m

Optical Path

Superior Mirage

The Green Flash

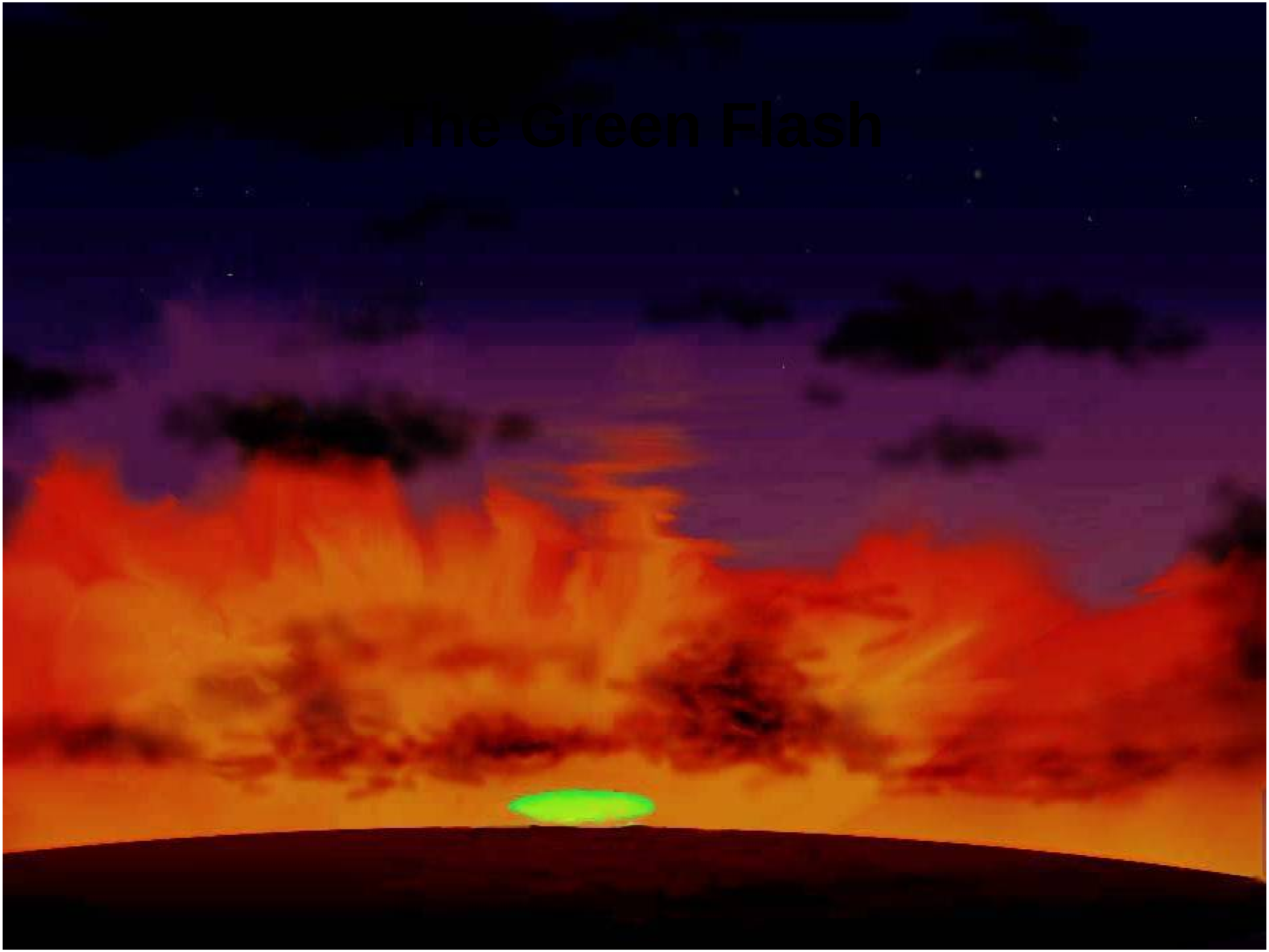
In 1700 several scientists reported the observation of a green flash just after sunset. The observations were made over the Tirrhenian sea. Newton had just discovered the complementarity of colours and gave a quick explanation of the green flash as an optical illusion.
The green flash was forgotten for about 50 years, until the observation of the more rare event of a green flash at sunrise over the Adriatic sea.

Now beautiful pictures of the green flash can be found on the web.

The Green Flash

The Green Flash

The Green Flash



The Green Flash

At sunset and sunrise the Sun has an upper green rim due to the combined effect of diffusion and attenuation of blue light. The green rim is normally too narrow to be seen without optical aid.

At the folding point of a mirage, there is a zone of sky, parallel to the horizon, in which strong vertical stretching occurs. This broadens the green rim into a feature wide enough to be seen.

Scintillations

The refractive index causes not only image distortions, but also intensity variations.

The presence of a variable atmosphere, because of either the turbulence of air or the movement of the line of sight, causes small movements of the line of sight.

These movements introduce a stretching and squeezing of the image, which in the case of a point source also generate intensity variations.

The scintallations are the intensity variations observed as a function of time for a point source.