# The Theoretical Problem of Partial Coherence and Partial Polarization in PolSAR and PolInSAR

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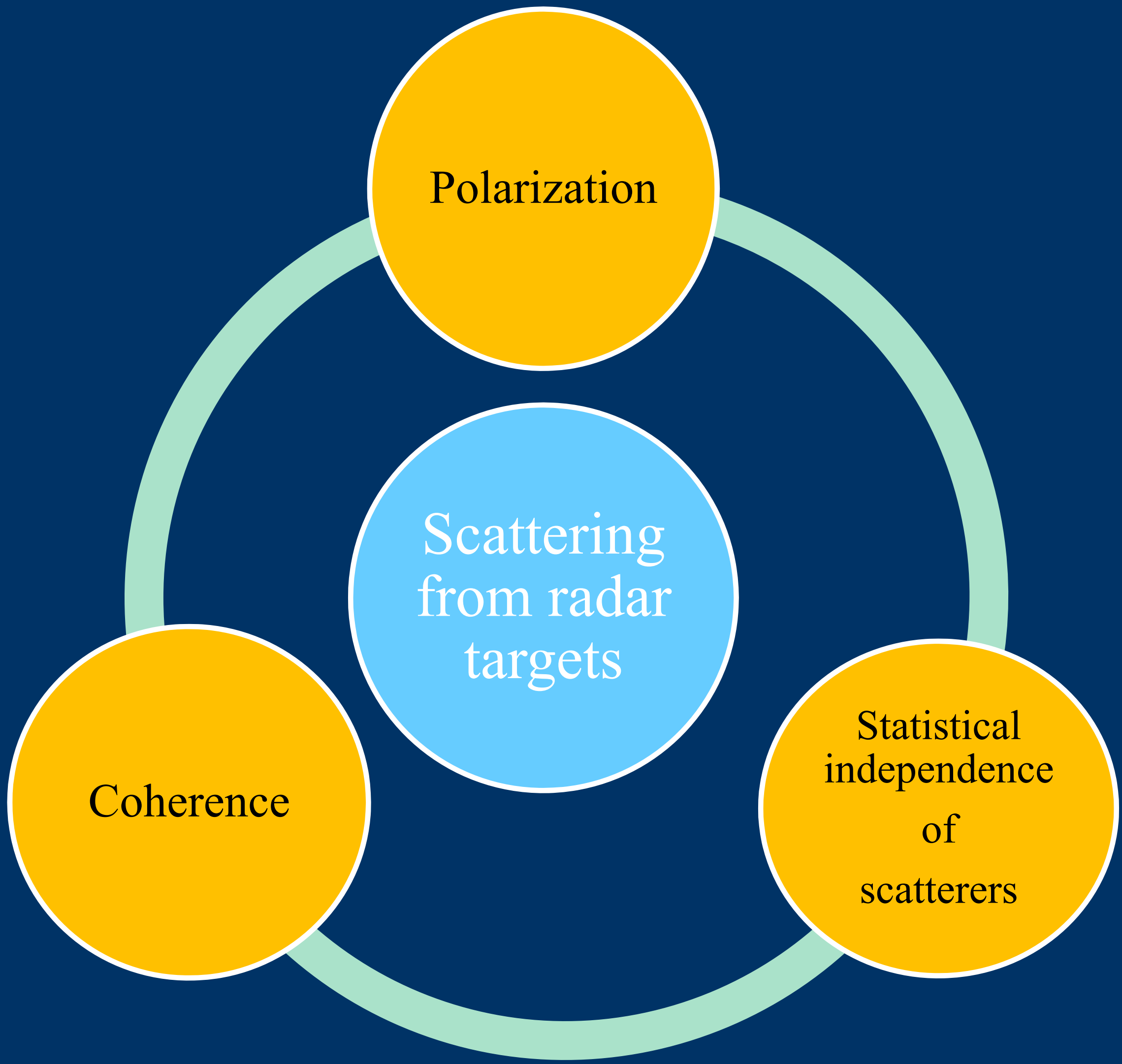
# Outline

1. Concepts of polarization, coherence and statistical independence in scattered fields.
2. Optics field-based view vs. SAR target-based view.
3. PolSAR and orthogonality
4. The coherence tensor and multidimensional tensor decompositions.
5. PolInSAR and the definition of coherence.
6. A numerical, computer-controlled scattering experiment over a rough surface.

# *General framework of study: polarization, coherence and statistical independence of scattering phenomena.*



# Optical paradigm in polarimetry

- "Despite the great deal of literature that exists about polarization of light and other random electromagnetic radiation, the underlying theory has hardly advanced since 1858 when G.G. Stokes introduced four parameters which now bear his name, to characterize the state of polarization of a light wave." (Wolf, 2003)

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}^{\text{scat}} = \mathbf{M} \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}^{\text{inc}} \qquad \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} |E_x|^2 + |E_y|^2 \\ |E_x|^2 - |E_y|^2 \\ 2\,\mathrm{Re}\{E_x\, E_y{}^*\} \\ 2\,\mathrm{Im}\{E_x\, E_y{}^*\} \end{bmatrix}$$

# *Optics paradigm in polarimetry*

- Wolf (1954), Parrent (1960) and Marathay (1963) set a path to follow, but they were mainly concerned with the characterization of the state of polarization of an electromagnetic field (Stokes vector) and not with the characterization of the target (Mueller matrix).

$$E_x(t) = a_x(t)\exp\{j[\phi_x(t) - 2\pi\bar{\omega}t]\}$$

$$E_y(t) = a_y(t)\exp\{j[\phi_y(t) - 2\pi\bar{\omega}t]\}$$

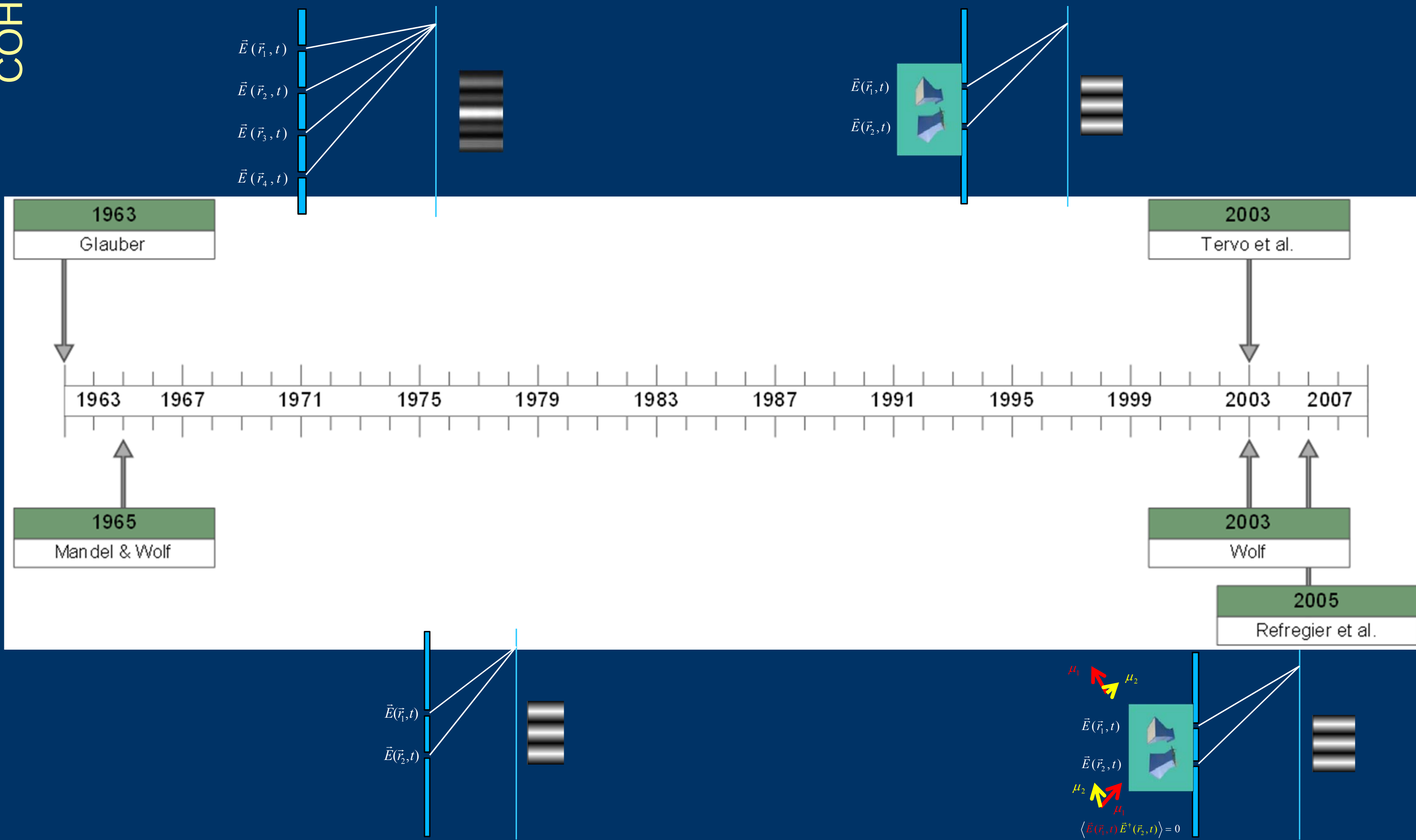
**Quasimonochromatic Electromagnetic field of Central frequency $\bar{w}$**

$$\mathbf{J} = \langle \mathbf{E} \otimes \mathbf{E}^{\dagger} \rangle = \begin{pmatrix} \langle E_x(t)E_x(t)^* \rangle & \langle E_x(t)E_y(t)^* \rangle \\ \langle E_y(t)E_x(t)^* \rangle & \langle E_y(t)E_y(t)^* \rangle \end{pmatrix} = \sum_{\alpha=0}^{3} S_\alpha \boldsymbol{\sigma}_\alpha \, ; S_\alpha = \mathrm{Tr}\{\boldsymbol{\sigma}_\alpha \mathbf{J}\}$$

$$\boldsymbol{\sigma}_\alpha = \text{Pauli matrices}$$

**2-D Field Coherency Matrix**

# Coherence definitions



# Glauber's concept of coherence

**N-th order correlation function**

$$G^{(n)}_{\mu_1,\ldots,\mu_{2n}}(\vec{r}_1,t_1;\ldots;\vec{r}_n,t_n;\vec{r}_{n+1},t_{n+1};\ldots;\vec{r}_{2n},t_{2n}) = \left\langle E^{\dagger}_{\mu_1}(\vec{r}_1,t_1)\cdots E^{\dagger}_{\mu_n}(\vec{r}_n,t_n)E_{\mu_{n+1}}(\vec{r}_{n+1},t_{n+1})\cdots E_{\mu_{2n}}(\vec{r}_{2n},t_{2n})\right\rangle$$

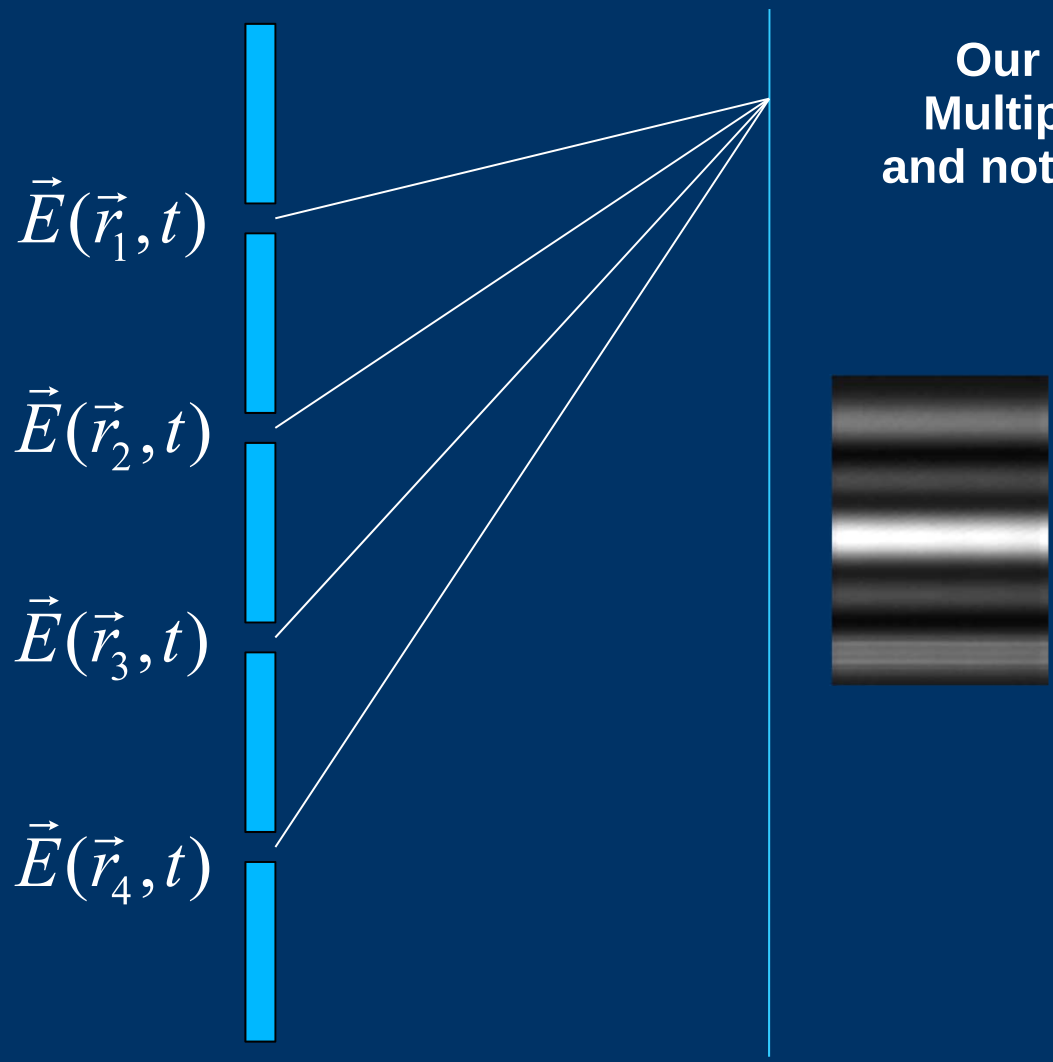
$$\exists\left\{\varepsilon_{\mu_j}(\vec{r}_j,t_j)\right\}_{j=1}^{2n} \,/\, G^{(n)}_{\mu_1,\ldots,\mu_{2n}}(\vec{r}_1,t_1;\ldots;\vec{r}_n,t_n;\vec{r}_{n+1},t_{n+1};\ldots;\vec{r}_{2n},t_{2n}) = \varepsilon^{\dagger}_{\mu_1}(\vec{r}_1,t_1)\cdots\varepsilon^{\dagger}_{\mu_n}(\vec{r}_n,t_n)\varepsilon_{\mu_{n+1}}(\vec{r}_{n+1},t_{n+1})\cdots\varepsilon_{\mu_{2n}}(\vec{r}_{2n},t_{2n})$$

**Glauber's definition of coherency**

In Glauber's definition full coherence implies full polarization

# *Coherence in a multiple slit interferometer (Glauber-like)*

**Our interference is Multiplicative (digital) and not Additive (analog)**



# Wolf's concept of coherence

**Cross spectral density matrix**

$$\mathbf{W}(\vec{r}_1,\vec{r}_2;w)=\begin{bmatrix}\left\langle E_x^*(\vec{r}_1,w)E_x(\vec{r}_2,w)\right\rangle & \left\langle E_x^*(\vec{r}_1,w)E_y(\vec{r}_2,w)\right\rangle \\ \left\langle E_y^*(\vec{r}_1,w)E_x(\vec{r}_2,w)\right\rangle & \left\langle E_y^*(\vec{r}_1,w)E_y(\vec{r}_2,w)\right\rangle\end{bmatrix}$$

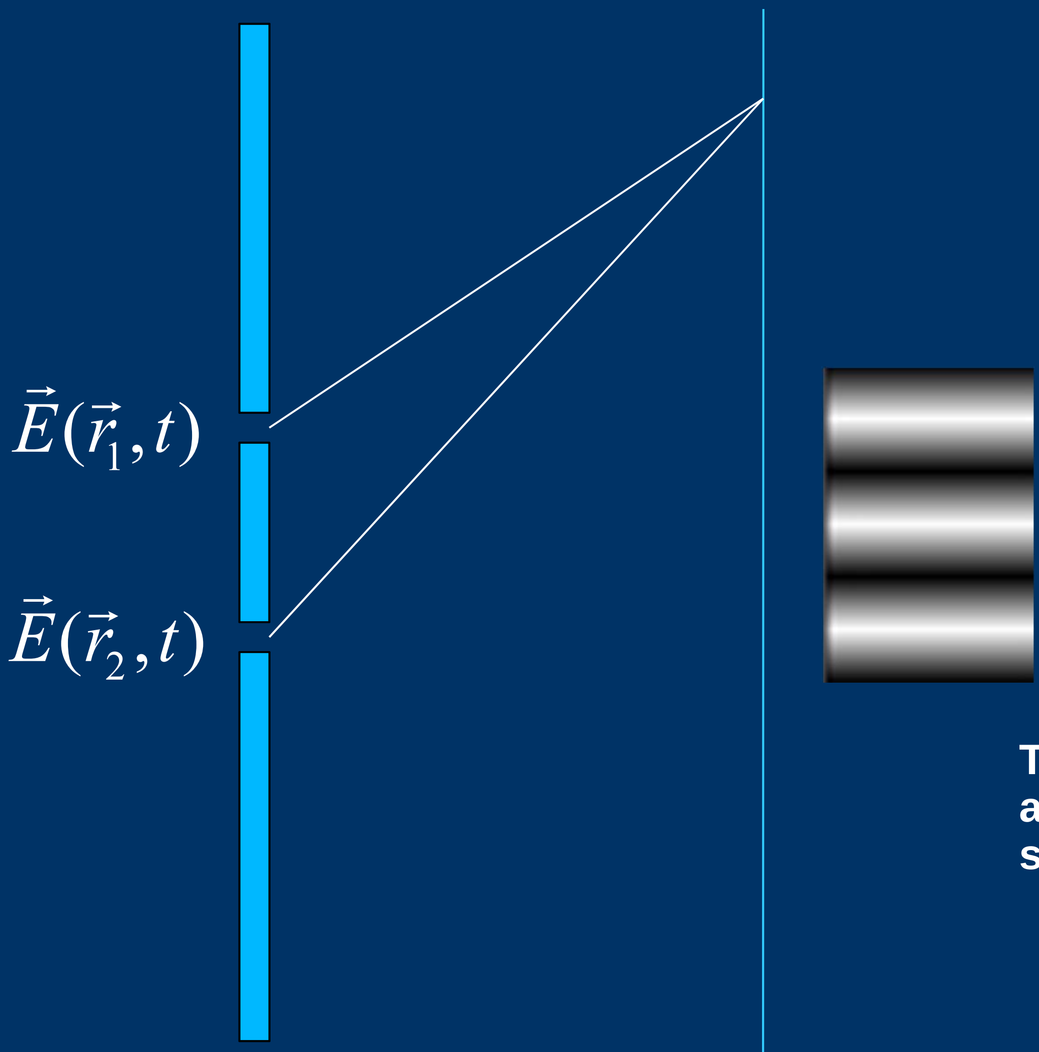
$$\mu(\vec{r}_1,\vec{r}_2;w)=\frac{\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_2;w)}{\sqrt{\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_1;w)}\sqrt{\mathrm{Tr}\,\mathbf{W}(\vec{r}_2,\vec{r}_2;w)}}$$

**Wolf's definition of degree of coherency**

$$P(w)=\sqrt{1-\frac{4\det\mathbf{W}(\vec{r}_1,\vec{r}_1;w)}{\left[\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_1;w)\right]^2}}$$

**Wolf's definition of degree of polarization**

# Coherence in a double slit interferometer (Wolf-like)




$$\text{What if } \mathbf{W}(\vec{r}_1,\vec{r}_2;w)=\frac{1}{2}\begin{bmatrix} I_0 & 0 \\ 0 & I_0 \end{bmatrix} ?$$

$$\mu = 1$$

$$P_1 = P_2 = 0$$

**Then, the field is completely unpolarized at each pinhole but it is completely spatially coherent at these points.**

# *Wolf's concept of coherence*

**Cross spectral density matrix**

$$\mathbf{W}(\vec{r}_1,\vec{r}_2;w)=\begin{bmatrix}\left\langle E_x^*(\vec{r}_1,w)E_x(\vec{r}_2,w)\right\rangle & \left\langle E_x^*(\vec{r}_1,w)E_y(\vec{r}_2,w)\right\rangle \\ \left\langle E_y^*(\vec{r}_1,w)E_x(\vec{r}_2,w)\right\rangle & \left\langle E_y^*(\vec{r}_1,w)E_y(\vec{r}_2,w)\right\rangle\end{bmatrix}$$

$$\mu(\vec{r}_1,\vec{r}_2;w)=\frac{\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_2;w)}{\sqrt{\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_1;w)}\sqrt{\mathrm{Tr}\,\mathbf{W}(\vec{r}_2,\vec{r}_2;w)}}$$

**Wolf's definition of degree of coherency**

$$P(w)=\sqrt{1-\frac{4\det\mathbf{W}(\vec{r}_1,\vec{r}_1;w)}{\left[\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_1;w)\right]^2}}$$

**Wolf's definition of degree of polarization**

Tervo and co-workers pointed out that μ is not invariant under position-dependent orthogonal transformations

# *Tervo's concept of coherence*

**Cross spectral density matrix**

$$\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)=\begin{bmatrix}\left\langle E_x^*(\vec{r}_1,\tau)E_x(\vec{r}_2,\tau)\right\rangle & \left\langle E_x^*(\vec{r}_1,\tau)E_y(\vec{r}_2,\tau)\right\rangle \\ \left\langle E_y^*(\vec{r}_1,\tau)E_x(\vec{r}_2,\tau)\right\rangle & \left\langle E_y^*(\vec{r}_1,\tau)E_y(\vec{r}_2,\tau)\right\rangle\end{bmatrix}$$

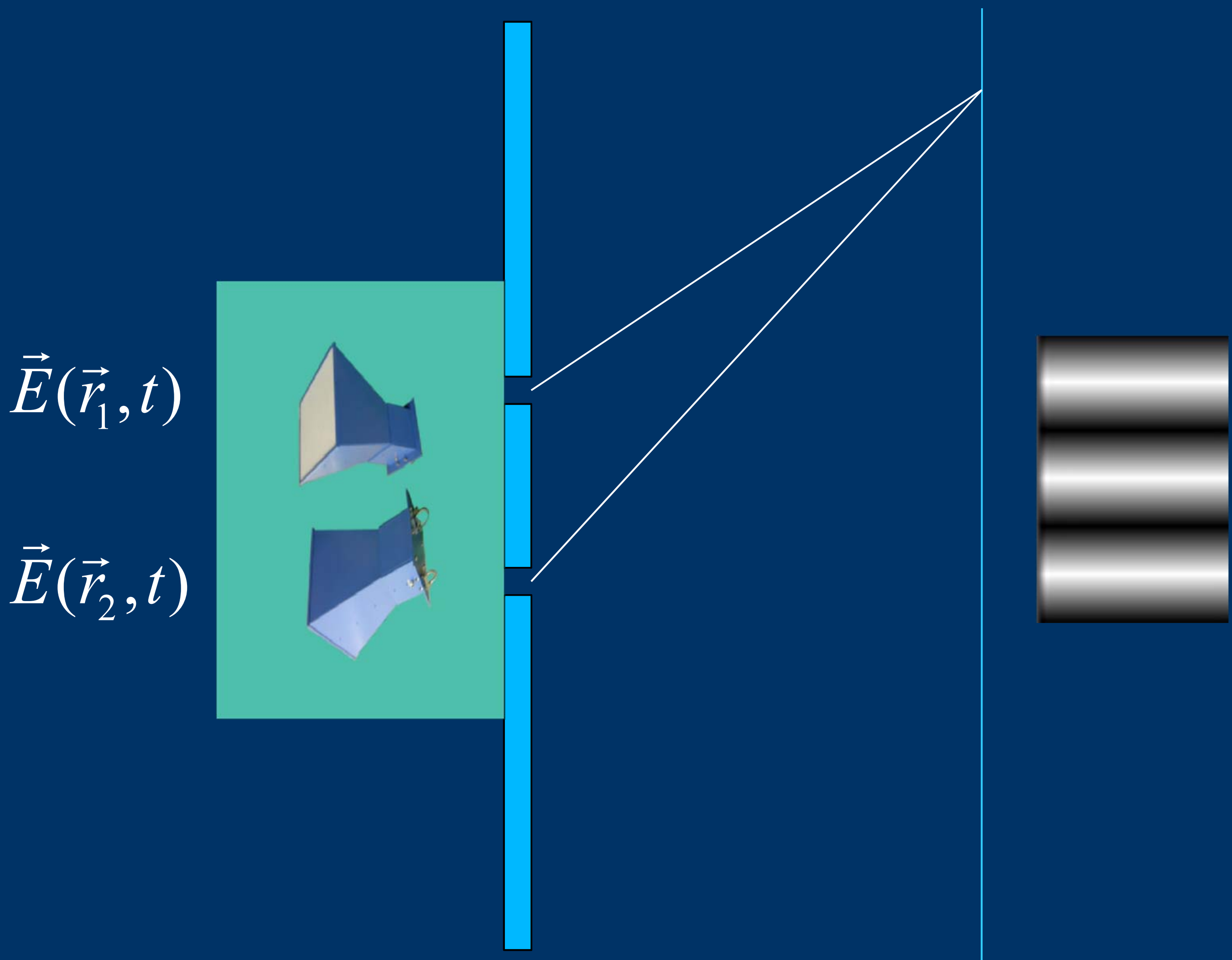
$$\tilde{\mu}^2(\vec{r}_1,\vec{r}_2;\tau)=\frac{\mathrm{Tr}\left\{\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)\mathbf{W}^\dagger(\vec{r}_1,\vec{r}_2;\tau)\right\}}{\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_1;\tau)\,\mathrm{Tr}\,\mathbf{W}(\vec{r}_2,\vec{r}_2;\tau)}$$

**Tervo's definition of degree of coherency**

$$P^2(\vec{r})=2\left[\tilde{\mu}^2(\vec{r},\vec{r};0)-\frac{1}{2}\right]$$

**Tervo's expression for the degree of polarization**

# *Coherence in a double slit interferometer (Tervo-like)*



# *Tervo's concept of coherence*

**Cross spectral density matrix**

$$\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)=\begin{bmatrix}\left\langle E_x^*(\vec{r}_1,\tau)E_x(\vec{r}_2,\tau)\right\rangle & \left\langle E_x^*(\vec{r}_1,\tau)E_y(\vec{r}_2,\tau)\right\rangle \\ \left\langle E_y^*(\vec{r}_1,\tau)E_x(\vec{r}_2,\tau)\right\rangle & \left\langle E_y^*(\vec{r}_1,\tau)E_y(\vec{r}_2,\tau)\right\rangle\end{bmatrix}$$

$$\tilde{\mu}^2(\vec{r}_1,\vec{r}_2;\tau)=\frac{\mathrm{Tr}\left\{\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)\mathbf{W}^\dagger(\vec{r}_1,\vec{r}_2;\tau)\right\}}{\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_1;\tau)\,\mathrm{Tr}\,\mathbf{W}(\vec{r}_2,\vec{r}_2;\tau)}$$

**Tervo's definition of degree of coherency**

$$P^2(\vec{r})=2\left[\tilde{\mu}^2(\vec{r},\vec{r};0)-\frac{1}{2}\right]$$

**Tervo's expression for the degree of polarization**

Refregier and Goudail found it not satisfactory that $\tilde{\mu}$ with $\mathbf{r_1}=\mathbf{r_2}$ produces the degree of polarization P

# Refregier 's concept of coherence

**Cross spectral density matrix**

$$\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)=\begin{bmatrix}\left\langle E_x^*(\vec{r}_1,\tau)E_x(\vec{r}_2,\tau)\right\rangle & \left\langle E_x^*(\vec{r}_1,\tau)E_y(\vec{r}_2,\tau)\right\rangle \\ \left\langle E_y^*(\vec{r}_1,\tau)E_x(\vec{r}_2,\tau)\right\rangle & \left\langle E_y^*(\vec{r}_1,\tau)E_y(\vec{r}_2,\tau)\right\rangle\end{bmatrix} \quad \mathbf{C}(\vec{r}_i;\tau)=\mathbf{W}(\vec{r}_i,\vec{r}_i;\tau);i=1,2$$

$$\vec{E}'(\vec{r}_i,\tau)=\mathbf{R}_i\,\vec{E}(\vec{r}_i,\tau);i=1,2 \xrightarrow{\textit{Equivalence class}} \mathscr{C}[\vec{E}'(\vec{r}_1,\tau),\vec{E}'(\vec{r}_2,\tau)]$$

$$\mathscr{C}[\mathbf{R}_1\,\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)\,\mathbf{R}_2^\dagger,\mathbf{R}_1\,\mathbf{C}(\vec{r}_1;\tau)\,\mathbf{R}_1^\dagger,\mathbf{R}_2\,\mathbf{C}(\vec{r}_2;\tau)\,\mathbf{R}_2^\dagger]$$

$$\downarrow \mathbf{R}_i=\mathbf{C}(\vec{r}_i;\tau)^{-1/2}$$

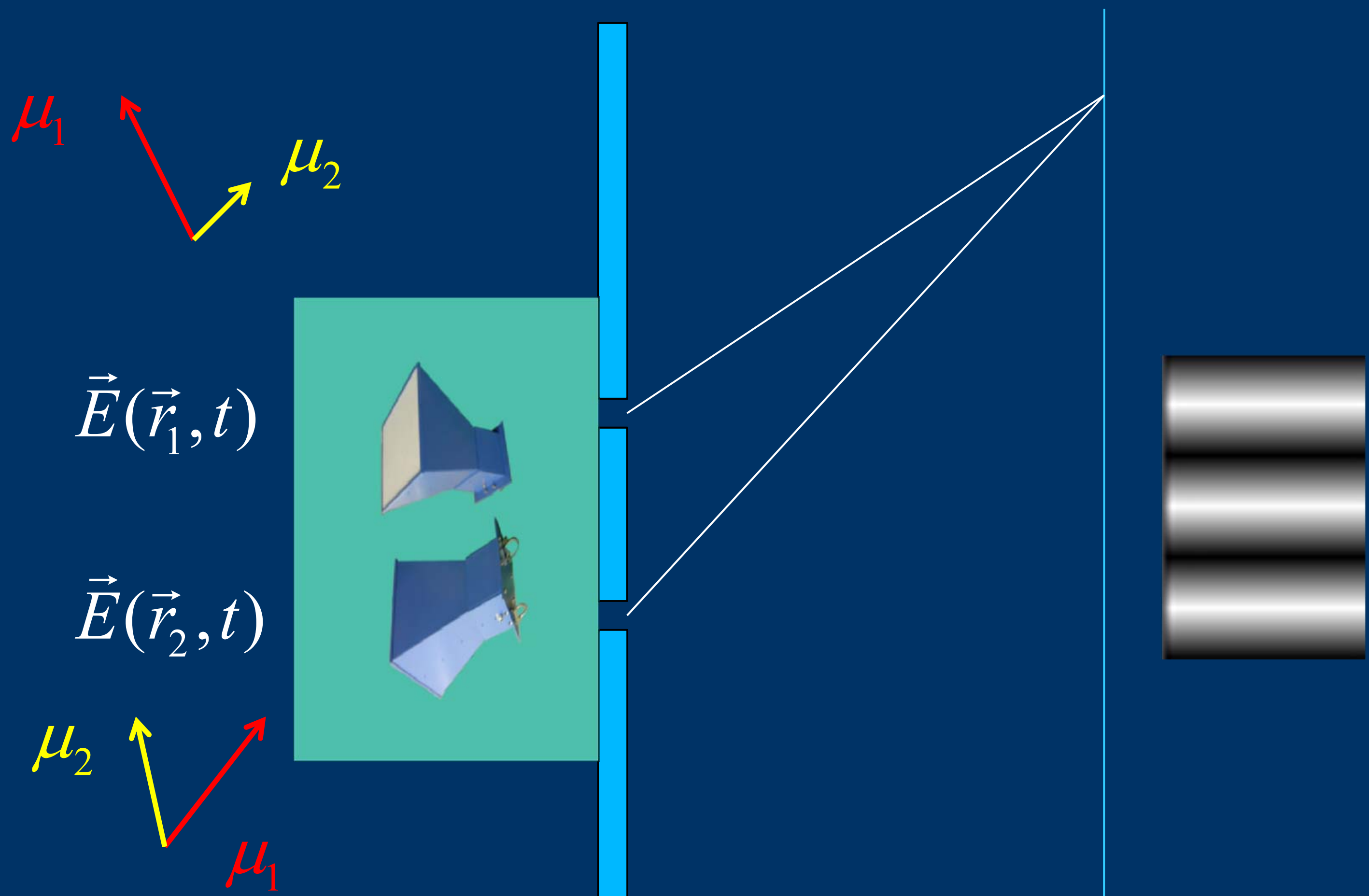
$$\left\{\mathbf{M}\equiv\mathbf{C}(\vec{r}_1;\tau)^{-1/2}\,\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)\,\mathbf{C}(\vec{r}_2;\tau)^{-1/2},\mathbf{I}_d,\mathbf{I}_d\right\}$$

$$\mathbf{M}=\mathbf{U}\begin{bmatrix}\mu_1 & 0\\ 0 & \mu_2\end{bmatrix}\mathbf{V}^\dagger$$

(**U**, **V** are unitary matrices)

Refregier proposed the use of $\mu_1$ and $\mu_2$ as intrinsic degrees of coherence and are invariant by local linear deterministic transformations that can modify the polarization properties.

# Coherence in a double slit interferometer (Refregier-like)




$$\left\langle \vec{E}(\vec{r}_1,t)\,\vec{E}^{\dagger}(\vec{r}_2,t) \right\rangle = 0$$

# Statistical Independence and Orthogonality

- Orthogonality and statistical independence are equivalent concepts at second order for two tuples of data $\{x_i\}$ and $\{y_i\}$ iff each has a null average:

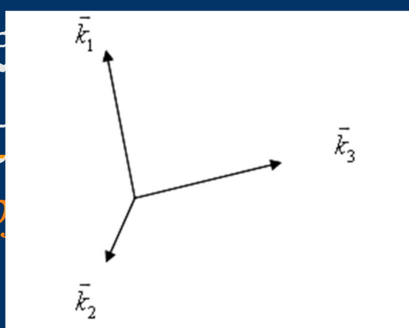
$$\langle x\, y \rangle = \sum_i x_i\, y_i = \begin{cases} \langle x \rangle \langle y \rangle & in\ general \\ 0 & if\ \ \langle x \rangle\ or \langle y \rangle = 0 \end{cases}$$

- Orthogonality as an indication of different components of a signal is at the base of harmonic methods such as MUSIC or ESPRIT
- Principal Component Analysis is also based on an orthogonal transformation of the data

# Eigenanalysis paradigm in polarimetry

# Optical paradigm in polarimetry

- Wolf (1954), Parrent (1960) and Marathay (1963) set a path to follow, but they were mainly concerned with the characterization of the state of polarization of an e…etic field (with Stokes vector) and not with t…rization of the target (Mueller matrix).




*Ob… Cloude and Pottier's SAR polarimetry* : $\mathbf{T} = \langle \mathbf{k}\,\mathbf{k}^{\dagger} \rangle;\ k_{\alpha} = \mathrm{Tr}\{\boldsymbol{\sigma}_{\alpha}\mathbf{S}\}$

$E_x(t) = a_x(t)\exp\{ j[\phi_x(t) - 2\pi\bar{\omega} t]\}$

$E_y(t) = a_y(t)\exp\{ j[\phi_y(t) - 2\pi\bar{\omega} t]\}$

**Quasimonochromatic Electromagnetic field of Central frequency $\bar{w}$**

*Cloude Decomposition* : $\mathbf{T} = \sum_{\mu=0}^{3} \lambda_{\mu} \vec{u}_{\mu} \vec{u}_{\mu}^{\dagger}$

*Orthogonality* : $\mathrm{Tr}\{\mathbf{L}_{\alpha}^{\dagger}\mathbf{L}_{\beta}\} = \vec{u}_{\alpha}^{\dagger} \cdot \vec{u}_{\beta} = \delta_{\alpha\beta} \Longrightarrow \vec{e}_1^{\dagger}\mathbf{L}_{\alpha}^{\dagger}\mathbf{L}_{\beta}\vec{e}_1 + \vec{e}_2^{\dagger}\mathbf{L}_{\alpha}^{\dagger}\mathbf{L}_{\beta}\vec{e}_2 = \delta_{\alpha\beta}$

$$\mathbf{J} = \langle \mathbf{E}\mathbf{E}^{\dagger} \rangle = \begin{pmatrix} \langle E_x(t)E_x(t)^* \rangle & \langle E_x(t)E_y(t)^* \rangle \\ \langle E_y(t)E_x(t)^* \rangle & \langle E_y(t)E_y(t)^* \rangle \end{pmatrix} = \sum_{\alpha=0}^{3} S_{\alpha}\boldsymbol{\sigma}_{\alpha};\ S_{\alpha} = \mathrm{Tr}\{\boldsymbol{\sigma}_{\alpha}\mathbf{J}\}$$

$\mathbf{M} = \sum_{\mu=0}^{3} \lambda_{\mu} \boldsymbol{\Lambda}^{\dagger}(\mathbf{L}_{\alpha} \otimes \mathbf{L}_{\alpha}^{\dagger})\boldsymbol{\Lambda}\ ;\ \Lambda_{\alpha\beta} = [\boldsymbol{\sigma}_{\beta}]_{nm}\ \ (\alpha = 2n + m)$

$\mathbf{L}_{\mu} = \sum_{\alpha=0}^{3} [\vec{u}_{\alpha}]_{\mu} \boldsymbol{\sigma}_{\mu}$

$\boldsymbol{\sigma}_{\alpha}$ = Pauli matrices

**2-D Field Coherency Matrix**

*Hilbert – Schmidt inner product* : $\langle \mathbf{A}, \mathbf{B} \rangle_{\mathrm{HS}} = \mathrm{Tr}\{\mathbf{A}^{\dagger}\mathbf{B}\} = \sum_{n} (\mathbf{A}\vec{e}_n)^{\dagger}\mathbf{B}\vec{e}_n = \sum_{n} \vec{e}_n^{\dagger}\mathbf{A}^{\dagger}\mathbf{B}\vec{e}_n,\ \forall\{\vec{e}_n\}$ orthonormal basis

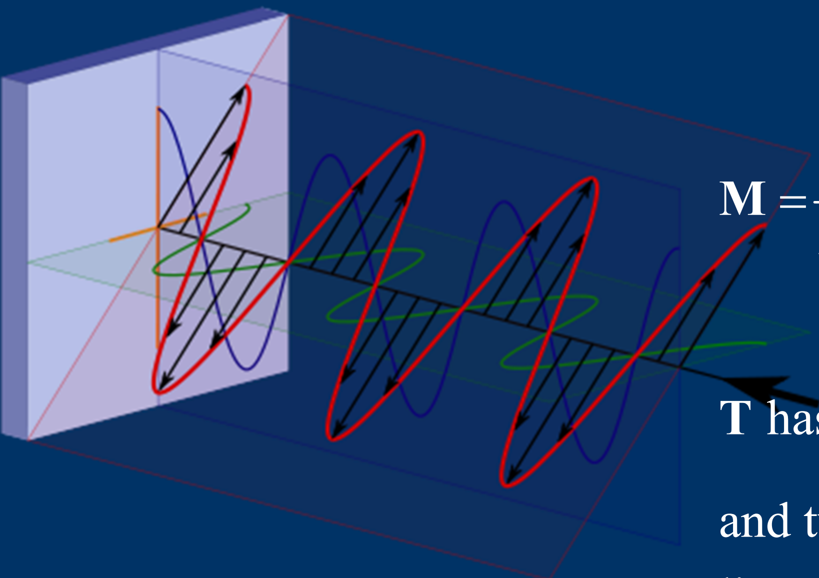
# Open issues in Eigenanalysis-based PolSAR paradigm

- All the work done on the definition of coherence, is based on field correlations that involve two or more points. As a consequence, partial coherence and partial polarization are two interrelated but different concepts and a current topic of research and it is not justified to use partial polarization to fully characterize partial coherence.
- The spectral decomposition of the target coherency matrix cannot be selected based on its uniqueness, because other incoherent decompositions are available.
- The orthogonality of the eigenvectors does not guarantee either a physically prevalent role for the spectral decomposition or a physical meaning for the eigenvectors themselves.
- For all these reasons, an eigenanalysis-based decomposition does not split the coherency matrix into different physical mechanisms in the most general case (more than one eigenvalue).

# Open issues in Eigenanalysis-based PolSAR paradigm

- Example:

A medium that behaves as a on the average, as the incoherent superposition of a horizontal linear polarizer + another medium that acts, on the average, as a linear polarizer at 45°.



$$\mathbf{M}=\frac{1}{2}\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \mathbf{T}=\frac{1}{4}\begin{bmatrix} 2 & -1+j & -1-j & 2 \\ -1-j & 2 & 0 & -1-j \\ -1+j & 0 & 2 & -1+j \\ 2 & -1+j & -1-j & 2 \end{bmatrix}$$

**T** has two non-null eigenvectors that verify $\lambda_{\max}/\lambda_{\min} = 3$

and two associated eigenvectors that do not correspond to the linear polarizers

# *HOW DOES ALL THIS AFFECT POLINSAR?*

# Field coherence vs. Target coherence

# PolInSAR's starting point

**Field interferometric covariance**

$$\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)=\left\langle \vec{E}(\vec{r}_1,\tau)\,\vec{E}(\vec{r}_1,\tau)^{\dagger}\right\rangle$$

**Wolf's**

$$\mu(\vec{r}_1,\vec{r}_2;w)=\frac{\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_2;w)}{\sqrt{\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_1;w)}\sqrt{\mathrm{Tr}\,\mathbf{W}(\vec{r}_2,\vec{r}_2;w)}}$$

**Tervo's**

$$\tilde{\mu}^2(\vec{r}_1,\vec{r}_2;\tau)=\frac{\mathrm{Tr}\left\{\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)\mathbf{W}^{\dagger}(\vec{r}_1,\vec{r}_2;\tau)\right\}}{\mathrm{Tr}\,\mathbf{W}(\vec{r}_1,\vec{r}_1;\tau)\,\mathrm{Tr}\,\mathbf{W}(\vec{r}_2,\vec{r}_2;\tau)}$$

**Refregier's**

$$\mathbf{W}(\vec{r}_1,\vec{r}_2;\tau)=\mathbf{U}'\begin{bmatrix}\mu_1 & 0\\ 0 & \mu_2\end{bmatrix}\mathbf{V}'^{\dagger}$$

**There are three different ways of looking at the SVD of a matrix:**

**1) A matrix decomposition scheme in three pieces**

**2) A summation of Kronecker vector products**

**3) The optimization of a Rayleigh quotient (variational approach)**

**Target interferometric covariance**

$$\mathbf{\Omega}_{12}(\vec{r}_1,\vec{r}_2;\tau)=\left\langle \vec{k}_1\,\vec{k}_2^{\dagger}\right\rangle$$

$$|\gamma|=\frac{\left|\left\langle \vec{w}_1^{\dagger}\,\mathbf{\Omega}_{12}\,\vec{w}_2\right\rangle\right|}{\sqrt{\left\langle \vec{w}_1^{\dagger}\,\mathbf{T}_{11}\,\vec{w}_1\right\rangle\left\langle \vec{w}_2^{\dagger}\,\mathbf{T}_{22}\,\vec{w}_2\right\rangle}};\quad \vec{w}_{1,2}\ \textit{arbitrary vectors}$$

$$\mathbf{\Omega}_{12}(\vec{r}_1,\vec{r}_2;\tau)=\mathbf{U}_{(3)}\begin{bmatrix}\gamma_1 & 0 & 0\\ 0 & \gamma_2 & 0\\ 0 & 0 & \gamma_3\end{bmatrix}\mathbf{V}_{(3)}^{\dagger}=\sum_{i=1}^{3}\gamma_i\,\vec{u}_i\,\vec{v}_i^{\dagger}$$

# Coherence in PolInSAR eigenanalysis

- PolInSAR eigenanalysis inherits the assumption of considering the scattering target vectors as faithful representatives of independent scattering phenomena and that is a problem from the point of view expressed in this work.

- However, it appears to offer a proper description of coherence through the singular values of the polarimetric interferometry matrix $\Omega_{12}$. Cloude and coworkers' PolInSAR paradigm typically contains three intrinsic degrees of coherence that are invariant under local linear deterministic transformations that can modify the polarization properties. We will show that this might also be a controversial issue.

# Coherence in PolInSAR eigenanalysis

Let us explore the meaning of the critical values of the Rayleigh quotient, that is, the singular values of $\Omega_{12}$!$\longrightarrow$ We go to higher dimensions

$$|\gamma| = \frac{\left|\left\langle \vec{w}_1^\dagger\, \mathbf{\Omega}_{12}\, \vec{w}_2 \right\rangle\right|}{\sqrt{\left\langle \vec{w}_1^\dagger\, \mathbf{T}_{11}\, \vec{w}_1 \right\rangle \left\langle \vec{w}_2^\dagger\, \mathbf{T}_{22}\, \vec{w}_2 \right\rangle}}; \quad \vec{w}_{1,2} \text{ arbitrary vectors}$$

or, if we pre-whiten the target scattering vectors

$$|\gamma| = \frac{\left|\left\langle \vec{w}_1^\dagger\, \Pi_{12}\, \vec{w}_2 \right\rangle\right|}{\sqrt{\left\langle \vec{w}_1^\dagger \vec{w}_1 \right\rangle \left\langle \vec{w}_2^\dagger\, \vec{w}_2 \right\rangle}}; \quad \vec{w}_{1,2} \text{ arbitrary vectors}$$

# Coherence tensor

**Glauber's N-th order correlation function**

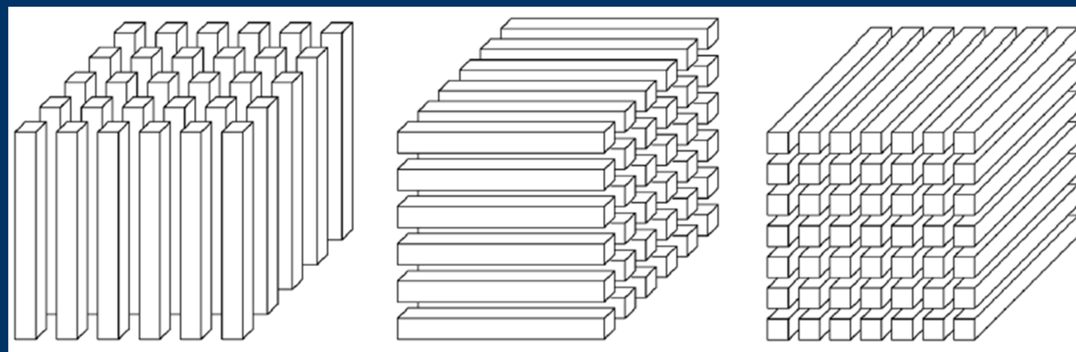
$$G^{(n)}_{\mu_1,\ldots,\mu_{2n}}(\vec{r}_1,t_1;\ldots;\vec{r}_n,t_n;\vec{r}_{n+1},t_{n+1};\ldots;\vec{r}_{2n},t_{2n}) = \left\langle E^{\dagger}_{\mu_1}(\vec{r}_1,t_1)\cdots E^{\dagger}_{\mu_n}(\vec{r}_n,t_n)E_{\mu_{n+1}}(\vec{r}_{n+1},t_{n+1})\cdots E_{\mu_{2n}}(\vec{r}_{2n},t_{2n})\right\rangle$$

$$\mathbf{T}^{\sigma_1,\ldots,\sigma_n}_{\sigma_{n+1},\ldots,\sigma_{2n}}(\vec{r}_1,t_1;\ldots;\vec{r}_n,t_n;\vec{r}_{n+1},t_{n+1};\ldots;\vec{r}_{2n},t_{2n}) = \left\langle k^{\sigma_1}(\vec{r}_1,t_1)\cdots k^{\sigma_n}(\vec{r}_n,t_n)k^{*}_{\sigma_{n+1}}(\vec{r}_{n+1},t_{n+1})\cdots k^{*}_{\sigma_{2n}}(\vec{r}_{2n},t_{2n})\right\rangle$$

**Coherence tensor**

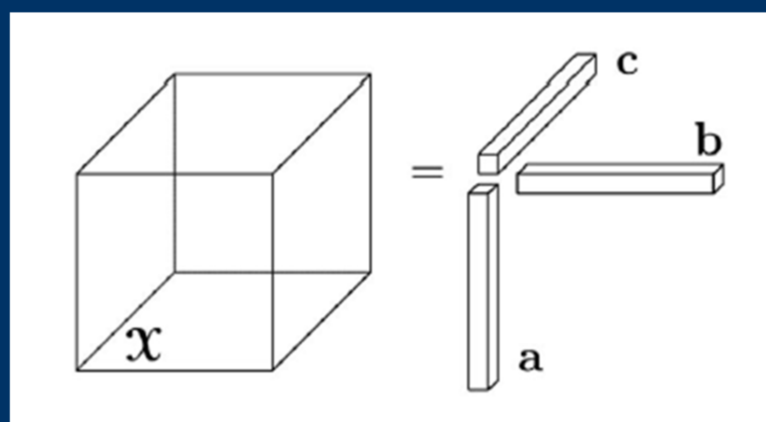
# *Decomposing a tensor: a one-slide course on tensors*

**A 3-D tensor**



**columns** **rows** **“tubes”** **=** ***fibers***

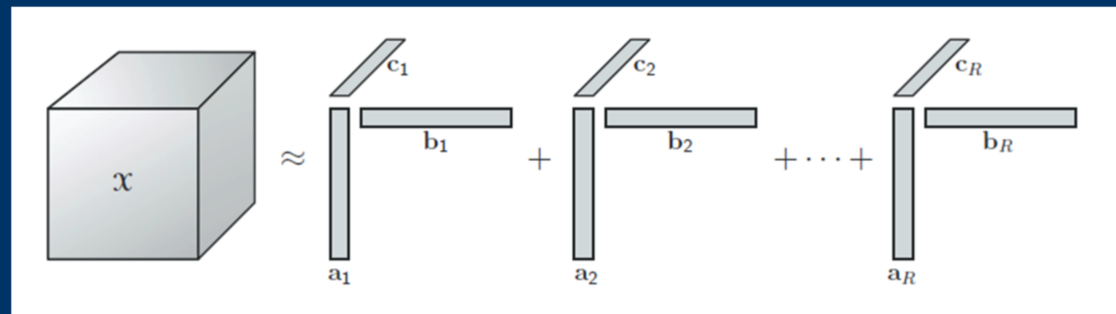
**A 3-D rank-one tensor**




**can be written as the outer (=Kronecker) product of three vectors**

# *Decomposing a tensor: we are looking for generalizations of the matrix SVD*

**1) The CANDECOMP/PARAFAC (CP) decomposition**




**2) Tucker decomposition**


a) ... matri... A, B and C ... ensor

b) The co... mid-tensor G is not superdiagonal

Singular values obtained ... manner

... not necessarily ... critical values ... repro... Tucker results in any manner

decomposition.

**3) Optimization of the generalized 2N-dimensional coherence**

b) Singular or critical points are not scale inv... en, if N≠1, inv... uaranteed.

$$\Gamma^{[k]}(\vec{r}_1,t_1;\dots,\vec{r}_n,t_n;\vec{r}_{n+1},t_{n+1};\dots;\vec{r}_{2n},t_{2n}) = \frac{|\mathbf{T}[\vec{w}^{(1)\dagger},\dots,\vec{w}^{(n)\dagger},\vec{w}^{(n+1)},\dots,\vec{w}^{(2n)}]|}{\left\langle \vec{w}_1^{\dagger}\mathbf{T_{11}}\vec{w}_1\right\rangle^{1/2}\cdots\left\langle \vec{w}_{2n}^{\dagger}\mathbf{T_{2n\,2n}}\vec{w}_{2n}\right\rangle^{1/2}}$$

# Conclusions on tensor decomposition theorems

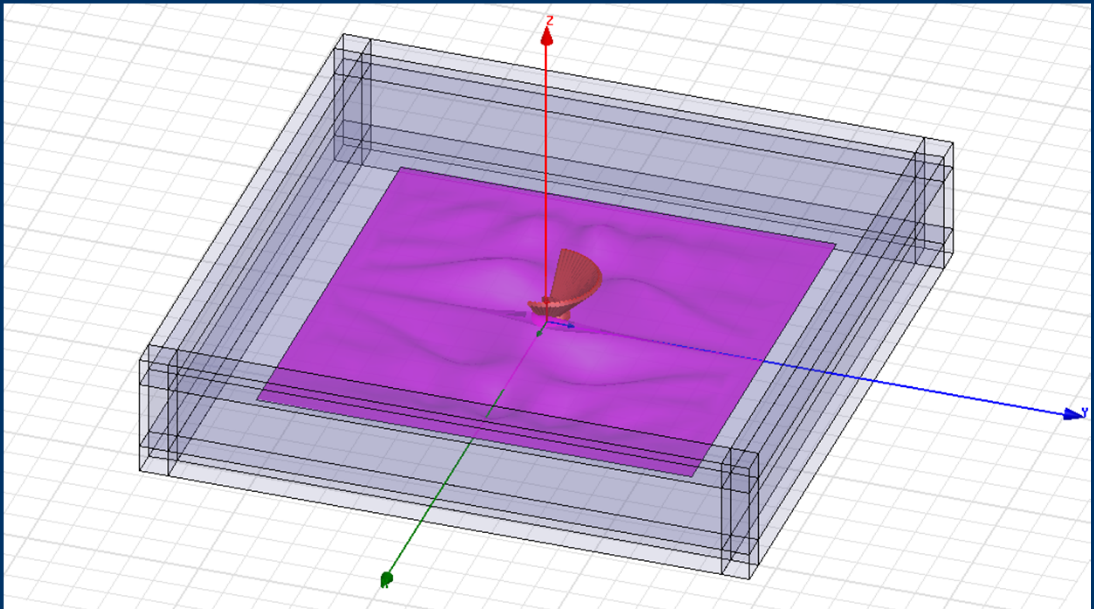
- The CP decomposition is only an algebraic version of the functional Glauber's condition; besides, it poses theoretical problems regarding uniqueness and truncation approximations.

- The Tucker decomposition does not produce superdiagonal core tensors, serves as a compression tool for the coherence tensor but it does not seem to provide a set of intrinsic degrees of coherence .

- The variational approach on the Rayleigh quotient is a promising approach to higher-than-two order coherence, even though it has some computational difficulties (it is an optimization problem).

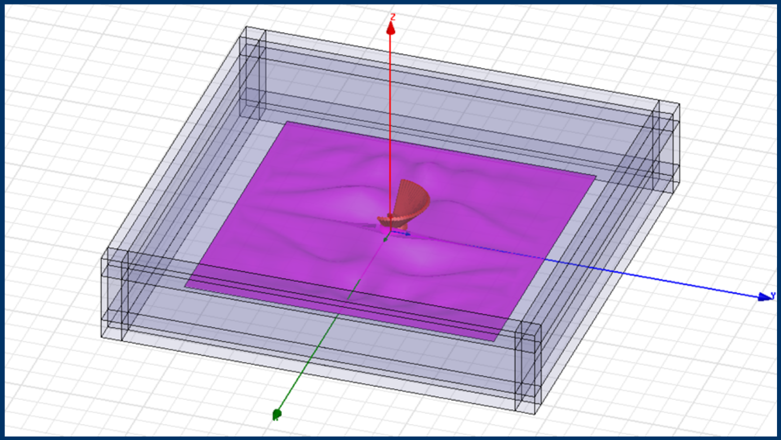
# Numerical experiment on the scattered field of a pec[1] rough surface

- The surface is rotated 35 times over 5º around axis *Z* to produce an ensemble of slightly rough surfaces of exponentially distributed height values that verify SPM conditions ( $k\,\sigma = 0.05$, $m = 0.2$)
- A 4-D coherence tensor was computed and analyzed at P1:[θ=45º,φ=-0.5 º], P2=[θ=45º,φ=0 º]; P3=[θ=44.5º,φ=0.5 º]; P4=[θ=45º,φ=0.5 º])
- The numerical simulation has been carried out with the Finite Element Method



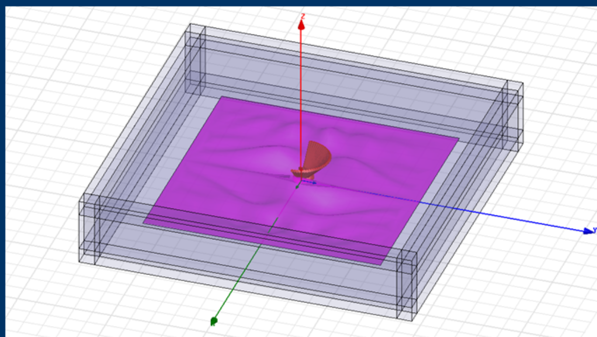
*PEC= Perfect Electric Conductor*

# Numerical experiment on the scattered field of a pec rough surface

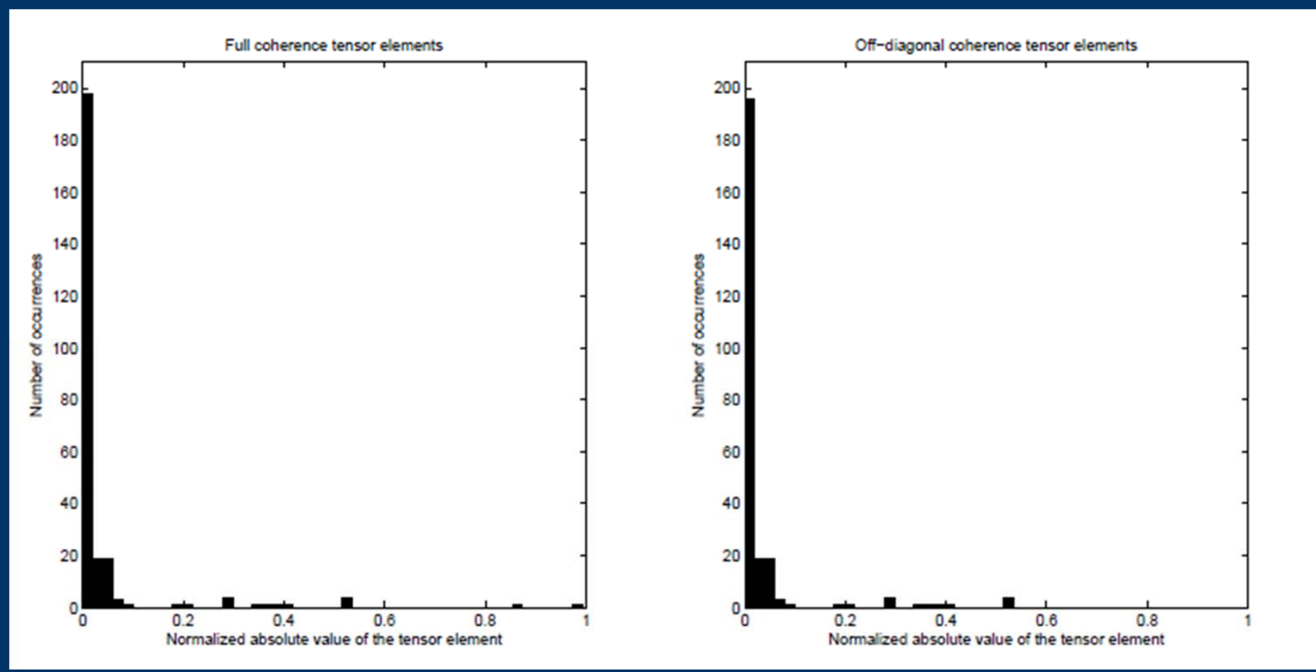


- CP decomposition → Tends to recover the 35 target scattering vectors making up the ensemble (CP-singular values all approximately equal)
- A Matlab toolbox for CP and Tucker decompositions is freely available from the Sandia Laboratories.

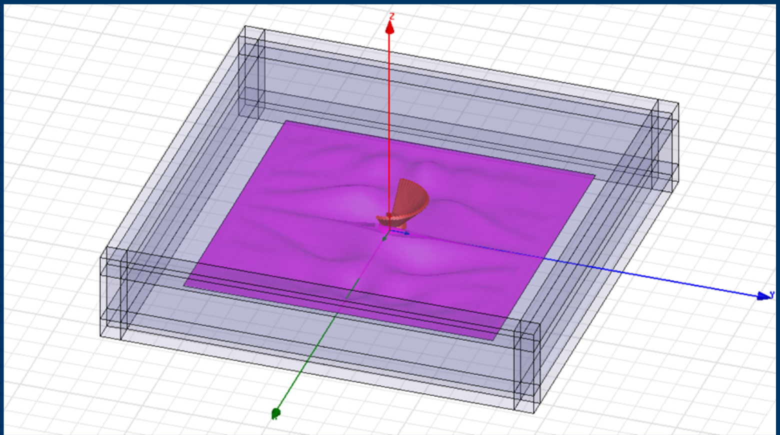
# Numerical experiment on the scattered field of a pec rough surface



- Tucker decomposition →



# Numerical experiment on the scattered field of a pec rough surface



- Variational approach → Four critical or singular values were found: (0.999, 0.972, 0.755, 0.719)

- On the other hand, the classical eigenanalysis-based approach produces:
  - Eigenvalues of $T_{11}$ equal to (0.6371, 0.3147, 0.0284, 0.0198)
  - Singular values of $\Pi_{12}$ are (0.999,0.998,0997,0.994)

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# Conclusions

- Polarization, coherence and statistical independence are related but different concepts. In particular, partial coherence and partial coherence should be described with different parameters and statistical independence is not an orthogonality condition in the space of target scattering vectors.
- There is a need for the definition of intrinsic degrees of coherence that may be satisfied by the singular values of $\Omega_{12}$ with a $\ell^2$ norm. However, going to higher-than-two dimensions and with the definition of an N-dimensional coherence tensor, there seems to be more detail in the description of coherence.
- Adequate scattering models, not based on the target scattering vector concept are needed to extract target characteristics from the analysis of coherence.

***Thank you for your attention and questions…***

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