Study of Speckle Noise Effects Over the Eigen Decomposition of Polarimetric SAR Data: A Review & Update

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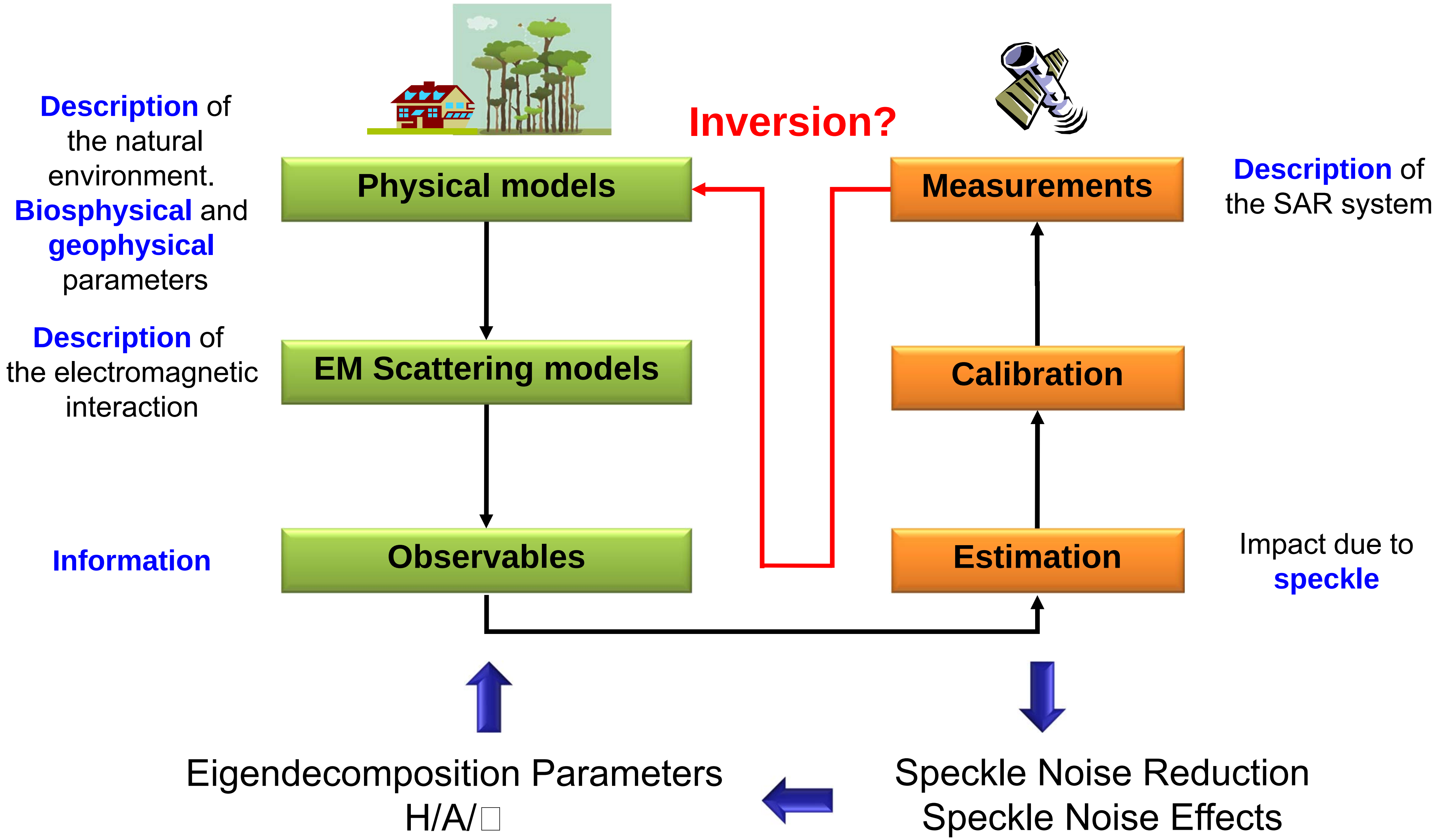
Quantitative Remote Sensing



State-of-the-Art

- Eigenvalues/Eigenvectors distribution analysis

  López-Martínez, C. & Pottier, E. Proc. POLINSAR 2005

  López-Martínez, C.; Pottier, E. & Cloude, S. *IEEE Trans. Geoscience and Remote Sensing,* 2005

  - Analytical analysis ➡ No limitations in terms of scatterer type
  - Joint eigenvalues pdf ➡ Complex analysis / Numerical analysis
    - Information about eigenvalues, H and A
  - Integration of the eigenvectors ➡ No information about eigenvectors

- Eigenvalues/Eigenvectors analysis based on simulated data

  Lee, J.; Ainsworth, T.; Kelly, J. & López-Martínez, C. *IEEE Trans. Geoscience and Remote Sensing,* 2008

  - Simulated data analysis ➡ Limitations in terms of scatterer type
  - Information about eigenvalues, H, A and □
  - No information about eigenvectors

- Indirect analysis based on the Touzi incoherent decomposition and the complex Jacobi rotations method

  Touzi, R., *Canadian Journal of Remote Sensing,* 2007

State-of-the-Art

Working hypothesis

$$\mathbf{k}=[S_1,S_2,\ldots,S_m]^T$$

$$S_k=\mathrm{N}_{C^2}\left(0,\sigma^2/2\right)$$

Complex Gaussian PDF

$$\mathbf{Z}_n=\frac{1}{n}\sum_{i=1}^{n}\mathbf{k}_i\mathbf{k}_i^H$$

$$\mathbf{Z}_n\sim \mathrm{W}(n,\mathbf{C})$$

Wishart PDF

$$p_{\mathbf{Z}_n}(\mathbf{Z}_n)=\frac{n^{mn}|\mathbf{Z}_n|^{n-m}}{|\mathbf{C}|^n\pi^m\prod_{i=1}^{m}\Gamma(n-i+1)}\mathrm{etr}\left(-n\mathbf{C}^{-1}\mathbf{Z}_n\right)$$

Limitations: $\mathbf{Z}_n$ $\mathbf{C}$ Positive definite; $n\geq m$

Joint sample eigenvalues distribution

$$p_{\hat{\lambda}_1,\hat{\lambda}_2,\ldots,\hat{\lambda}_m}\left(\hat{\lambda}_1,\hat{\lambda}_2,\ldots,\hat{\lambda}_m\right)=K(m,n,\lambda_1,\ldots,\lambda_m)\prod_{i=1}^{m}\hat{\lambda}_i^{n-m}\prod_{i<j}^{m}\left(\hat{\lambda}_i-\hat{\lambda}_j\right)\sum_{\pi\in S_m}\mathrm{sgn}(\pi)\prod_{i=1}^{m}\exp\left(-n\frac{\hat{\lambda}_i}{\lambda_{\pi_i}}\right)$$

Sorted sample eigenvalues $\infty>\hat{\lambda}_1\geq\hat{\lambda}_2\geq\cdots\geq\hat{\lambda}_m\geq 0$

$$K(m,n,\lambda_1,\ldots,\lambda_m)=\frac{\pi^{m(m-1)}n^{\frac{m}{2}(2n-m+1)}}{\tilde{\Gamma}_m(m)\tilde{\Gamma}_m(n)}\frac{\prod_{k=1}^{m-1}k^{m-k}}{\prod_{i=1}^{m}\lambda_i^n\prod_{i<j}^{m}\left(\lambda_j^{-1}-\lambda_i^{-1}\right)}$$

- **First eigenvalue** overestimated, **second** over/underestimated, **third** underestimated
- **Entropy** underestimated
- **Anisotropy** over/underestimated

Perturbation Analysis

Consider speckle noise from a perspective of a perturbation of the covariance/coherency matrices

$$\hat{\mathbf{T}} = \mathbf{T} + \mathbf{E}$$

Krim, H.; Forster, P. & Proakis, J. *Signal Processing, IEEE Transactions on,* 1992

López-Martínez, C. & Fàbregas, X. *IEEE Trans. Geoscience and Remote Sensing,* 2003

Quantification of the error matrix effect

$$\hat{\mathbf{T}} = \mathbf{T} + \varepsilon\mathbf{B} \qquad \mathbf{B} = \frac{\mathbf{E}}{\varepsilon} \qquad \varepsilon = \|\mathbf{E}\|$$

The goal is to relate the perturbation of the coherency matrix with a perturbation of the eigenvalues and eigenvectors. For □ sufficiently small

$$\hat{\lambda}_i(\varepsilon) = \lambda_i + k_{i1}\varepsilon + k_{i2}\varepsilon^2 + \ldots$$

$$\hat{\mathbf{u}}_i(\varepsilon) = \mathbf{u}_i + \sum_{k=1,k\neq i}^{m} \left(\varepsilon t_{ik1} + \varepsilon^2 t_{ik2} + \ldots\right)\mathbf{u}_k$$

For $\varepsilon \neq 0$ $\hat{\mathbf{u}}_i(\varepsilon)$ needs to be normalized

$$\mathbf{T}\mathbf{u} = \lambda\mathbf{u} \quad \Rightarrow \quad (\mathbf{T} + \varepsilon\mathbf{B})\hat{\mathbf{u}}(\varepsilon) = \hat{\lambda}_i(\varepsilon)\hat{\mathbf{u}}(\varepsilon)$$

First and Second Order Perturbation

The different orders are obtained by equating all the terms in □ or □^{2}

- First order analysis

$$\hat{\lambda}_i(\varepsilon) = \lambda_i + \varepsilon\beta_{ii} + O(\varepsilon^2)$$

$$\beta_{ij} = \mathbf{u}_i^H \mathbf{B} \mathbf{u}$$

$$\hat{\mathbf{u}}_i(\varepsilon) = \mathbf{u}_i + \varepsilon \sum_{k=1,k\neq i}^{m} \frac{\beta_{ki}}{\lambda_i - \lambda_k}\mathbf{u}_k + O(\varepsilon^2)$$

These terms contain the statistics associated to speckle

- Second order analysis

$$\hat{\lambda}_i(\varepsilon) = \lambda_i + \varepsilon\beta_{ii} + \varepsilon^2 \sum_{k=1,k\neq i}^{m} \frac{|\beta_{ik}|^2}{\lambda_i - \lambda_k} + O(\varepsilon^3)$$

$$\hat{\mathbf{u}}_i(\varepsilon) = \mathbf{u}_i + \varepsilon \sum_{k=1,k\neq i}^{m} \frac{\beta_{ki}}{\lambda_i - \lambda_k}\mathbf{u}_k + \varepsilon^2 \sum_{k=1,k\neq i}^{m} \left(\frac{\left(\sum_{j=1,j\neq i}^{m} \frac{\beta_{ji}\beta_{kj}}{\lambda_i - \lambda_j}\right) - \frac{\beta_{ii}\beta_{ki}}{\lambda_i - \lambda_k}}{\lambda_i - \lambda_k}\mathbf{u}_k \right) + O(\varepsilon^3)$$

This expression can be easily normalized

Higher Order Analysis

Necessary to have a more detailed characterization of the previous expressions

- Analysis of the asymptotic behavior
- Information for very close eigenvalues

- Third order analysis
  - Eigenvalue coefficients

$$k_{i3} = \sum_{k=1,k\neq i}^{m} \left(\frac{\left(\sum_{j=1,j\neq i}^{m} \frac{\beta_{ji}\beta_{kj}\beta_{ik}}{(\lambda_i - \lambda_j)s_j} \right) - \frac{\beta_{ii}\beta_{ki}\beta_{ik}}{(\lambda_i - \lambda_k)s_i}}{(\lambda_i - \lambda_k)s_k} \right)$$

  - Eigenvectors coefficients

$$t_{ik3} = \frac{\sum_{k=1,k\neq i}^{m} t_{ij2}\beta_{kj} - k_{i1}t_{ik2} - k_{i2}t_{ik1}}{\lambda_i - \lambda_k}$$

The order is associated to the product of □ terms

- Fourth order analysis
  - Eigenvalue coefficients

$$k_{i4} = \sum_{k=1,k\neq i}^{m} \frac{\left(\sum_{j=1,j\neq i}^{m} \frac{\beta_{ki}\beta_{jk}\beta_{kj}\beta_{ik}}{\lambda_i - \lambda_j} \right) - \frac{\beta_{ki}\beta_{ik}\beta_{ki}\beta_{ik}}{\lambda_i - \lambda_k}}{(\lambda_i - \lambda_k)^2}$$

Statisitics of the Perturbation Analysis

Introduction of the statistical information in the perturbation analysis

- The □ terms contain the information about speckle distribution
- The □ parameter contains the information about the filtering strength, i.e., the number of looks
- Re-parameterization

$$\zeta_{ij} = \varepsilon\beta_{ij} = \mathbf{u}_i^H \varepsilon \mathbf{B} \mathbf{u}$$

Considering the Gaussian hypothesis

- First order

$$E\left\{\zeta_{ii}\right\} = \mathbf{u}_i^H E\left\{\varepsilon \mathbf{B}\right\} \mathbf{u} = 0$$

→ Need of second order

- Second order

$$E\left\{\zeta_{ki}\zeta_{ik}\right\} = \frac{1}{n}\lambda_i\lambda_k$$

$$E\left\{\zeta_{ji}\zeta_{kj}\right\} = \frac{1}{n}\lambda_i\lambda_j tr\left(\mathbf{u}_i\mathbf{u}_k^H\right)$$

→ $\varepsilon = \frac{1}{\sqrt{n}}$

- Third order

$$E\left\{\zeta_{kl}\zeta_{mn}\zeta_{op}\right\} = \frac{1}{n^2} tr\left(\mathbf{u}_i\mathbf{u}_m^H\mathbf{T}\right) tr\left(\mathbf{u}_n\mathbf{u}_o^H\mathbf{T}\right)\mathbf{T} + \frac{1}{n^2}\mathbf{T}\mathbf{u}_l\mathbf{u}_m^H\mathbf{T}\mathbf{u}_n\mathbf{u}_o^H\mathbf{T}$$

→ Terms in $\frac{1}{n^2}$

Eigenvalues Characterization

Eigenvalues characterization based on the second order approximation expression

- Previous expression

  López-Martínez, C.; Pottier, E. & Cloude, S. *IEEE Trans. Geoscience and Remote Sensing*, 2005

$$\hat{\lambda}_i = \lambda_i + \frac{1}{n}\sum_{k=1,k\neq i}^{m} \frac{\lambda_i \lambda_k}{\lambda_i - \lambda_k} + O\left(n^{-1}\right)$$

- Current expression
  - Mean value

$$E\left\{\hat{\lambda}_i(\varepsilon)\right\} = \lambda_i + E\left\{\varepsilon\beta_{ii}\right\} + \sum_{k=1,k\neq i}^{m} \frac{E\left\{\varepsilon^2 \left|\beta_{ik}\right|^2\right\}}{\lambda_i - \lambda_k} + O\left(\varepsilon^3\right)$$

$$\hat{\lambda}_i = E\left\{\hat{\lambda}_i(\varepsilon)\right\} = \lambda_i + \frac{1}{n}\sum_{k=1,k\neq i}^{m} \frac{\lambda_i \lambda_k}{\lambda_i - \lambda_k} + O\left(n^{-2}\right)$$

  - Variance value

$$\operatorname{var}\left\{\hat{\lambda}_i(\varepsilon)\right\} = E\left\{\varepsilon^2 \beta_{ii}\beta_{ii}^*\right\} + O\left(\varepsilon^3\right)$$

$$\operatorname{var}\left\{\hat{\lambda}_i(\varepsilon)\right\} = \frac{1}{n}\lambda_i^2 + O\left(n^{-2}\right)$$

Eigenvectors Characterization

Eigenvectors characterization based on the second order approximation expression

- Mean value
  - Non-normalized expression

$$E\left\{\hat{\mathbf{u}}_i(\varepsilon)\right\} = \mathbf{u}_i + \sum_{k=1,k\neq i}^{m} \frac{E\left\{\varepsilon\beta_{ki}\right\}}{\lambda_i-\lambda_k}\mathbf{u}_k + \sum_{k=1,k\neq i}^{m}\left(\frac{\left(\sum_{j=1,j\neq i}^{m}\frac{E\left\{\varepsilon^2\beta_{ji}\beta_{kj}\right\}}{\lambda_i-\lambda_j}\right)-\frac{E\left\{\varepsilon^2\beta_{ii}\beta_{ki}\right\}}{\lambda_i-\lambda_k}}{\lambda_i-\lambda_k}\mathbf{u}_k\right)+O\left(\varepsilon^3\right)$$

$$\hat{\mathbf{u}}_i = \mathbf{u}_i + \frac{1}{n}\sum_{k=1,k\neq i}^{m}\frac{\lambda_i\lambda_k}{\left(\lambda_i-\lambda_k\right)^2}\mathbf{u}_k + O\left(n^{-2}\right)$$

  - Normalized expression

$$\hat{\hat{\mathbf{u}}}_i = \left(1-\frac{1}{2n}\sum_{k=1,k\neq i}^{m}\frac{\lambda_i\lambda_k}{\left(\lambda_i-\lambda_k\right)^2}\right)\mathbf{u}_i + \frac{1}{n}\sum_{k=1,k\neq i}^{m}\frac{\lambda_i\lambda_k}{\left(\lambda_i-\lambda_k\right)^2}\mathbf{u}_k + O\left(n^{-2}\right)$$

- Covariance matrix

$$\operatorname{cov}\left\{\hat{\hat{\mathbf{u}}}_i(\varepsilon)\right\} = \frac{1}{n}\sum_{k=1,k\neq i}^{m}\frac{\lambda_i\lambda_k}{\left(\lambda_i-\lambda_k\right)^2}\mathbf{T}_k + O\left(n^{-2}\right)$$

Entropy and Anisotropy Characterization

Entropy/Anisotropy characterization based on the second order approximation expression

- Perturbation analysis avoids the individual integration of eigenvalues if the joint pdf is considered
- Entropy

$$\hat{H} = H - \sum_{k=1,k\neq i}^{m} \frac{p_i}{n\ln(3)} \left\{ \frac{1}{2} + \left(1+\ln(p_i)\right) \sum_{k=1,k\neq i}^{m} \frac{\lambda_k}{\lambda_i - \lambda_k} \right\} + O\left(n^{-2}\right)$$

- Anisotropy

$$k_s = \sum_{k=1,k\neq 2}^{m} \frac{\lambda_2\lambda_k}{\lambda_2 - \lambda_k} + \sum_{k=1,k\neq 3}^{m} \frac{\lambda_3\lambda_k}{\lambda_3 - \lambda_k} \qquad k_d = \sum_{k=1,k\neq 2}^{m} \frac{\lambda_2\lambda_k}{\lambda_2 - \lambda_k} - \sum_{k=1,k\neq 3}^{m} \frac{\lambda_3\lambda_k}{\lambda_3 - \lambda_k}$$

$$\hat{A} = A - \frac{1}{n}\frac{\lambda_2 - \lambda_3}{\left(\lambda_2+\lambda_3\right)^2} k_s + \frac{1}{n}\frac{\lambda_2 - \lambda_3}{\left(\lambda_2+\lambda_3\right)^2}\left(\lambda_2^2 + \lambda_3^2\right) - \frac{1}{n}\frac{\lambda_2^2 - \lambda_3^2}{\left(\lambda_2+\lambda_3\right)^2} + \frac{1}{n}\frac{1}{\left(\lambda_2+\lambda_3\right)} k_d + O\left(n^{-2}\right)$$

Alpha Angles Characterization

$\square_i$ characterization based on the second order approximation expression

- Study from the first component of the normalized eigenvector

$$\hat{\alpha}_i = \arccos\left(\left|\hat{\mathbf{u}}_i(1)\right|\right) = \alpha_i - \frac{1}{2n}\left(\left|\mathbf{u}_i(1)\right| + \frac{\left|\mathbf{u}_i(1)\right|^3}{2}\right)\sum_{k=1,k\neq i}^{m}\frac{\lambda_i\lambda_k}{\left(\lambda_i-\lambda_k\right)^2}\left(\frac{2\Re\left\{\mathbf{u}_i(1)\mathbf{u}_k^*(1)\right\} - \left|\mathbf{u}_i(1)\right|^2}{\left|\mathbf{u}_i(1)\right|^2}\right) + O\left(n^{-2}\right)$$

$\bar{\alpha}$ characterization based on the second order approximation expression

- Use of the same procedure employed to derive the Entropy

$$\hat{\bar{\alpha}} = \bar{\alpha} - \sum_{i=1}^{3}\frac{p_i}{2n}\left(\left|\mathbf{u}_i(1)\right| + \frac{\left|\mathbf{u}_i(1)\right|^3}{2}\right) + \sum_{k=1,k\neq i}^{m}\frac{\lambda_i\lambda_k}{\left(\lambda_i-\lambda_k\right)^2}\left(\frac{2\Re\left\{\mathbf{u}_i(1)\mathbf{u}_k^*(1)\right\} - \left|\mathbf{u}_i(1)\right|^2}{\left|\mathbf{u}_i(1)\right|^2}\right) + O\left(n^{-2}\right)$$

Eigenvalues and derived parameters	Eigenvectors and derived parameters
$\frac{1}{\lambda_i-\lambda_k}$	$\frac{1}{\left(\lambda_i-\lambda_k\right)^2} \quad \left|\mathbf{u}_i(1)\right|^2$

Larger speckle noise effects

Simulation procedure

- Data simulated according to the complex Gaussian hypothesis, i.e., fully developed speckle
- Simulation of the true coherency matrix in terms of eigenvalues and eigenvectors
  - Eigenvalues considered relative to λ_1 to allow a correct representation, i.e., $\lambda_1=1$, λ_2/λ_1, λ_3/λ_1
  - Eigenvectors

$$\mathbf{u}_i = e^{j\phi_i}\left[\cos\alpha_i \quad \sin\alpha_i \sin\beta_i e^{j\delta_i} \quad \sin\alpha_i \cos\beta_i e^{j\gamma_i}\right]$$

The parameters are not independent as eigenvectors are orthonormal

α_1 Fixed

$\alpha_i \quad \beta_i \quad \delta_i \quad \phi_i \quad \gamma_i$ Variable to generate orthonal eigenvectors

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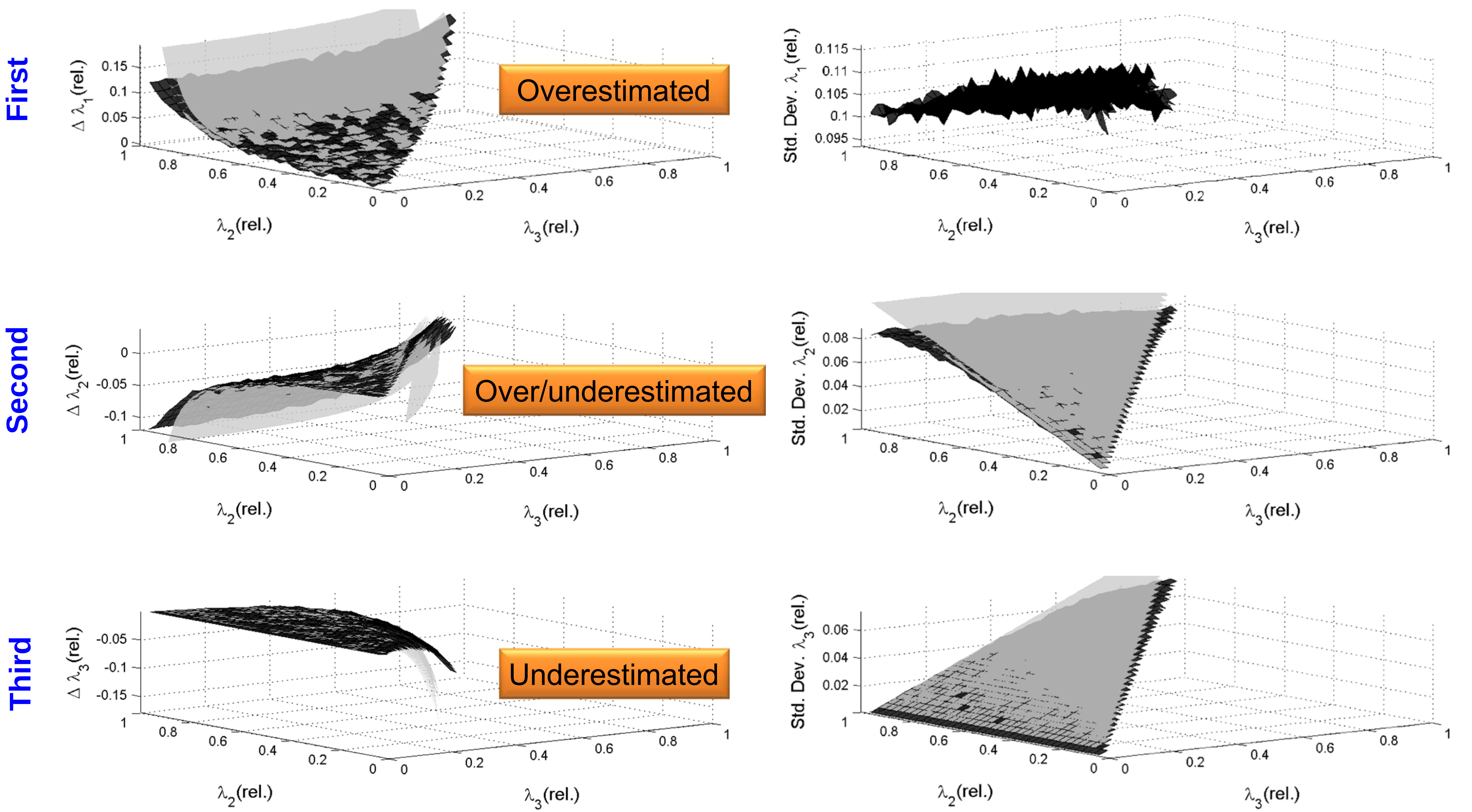
Evalutation in Terms of Simulated Data

Eigenvalues bias | **Eigenvalues Std. Dev.**



Simulated data

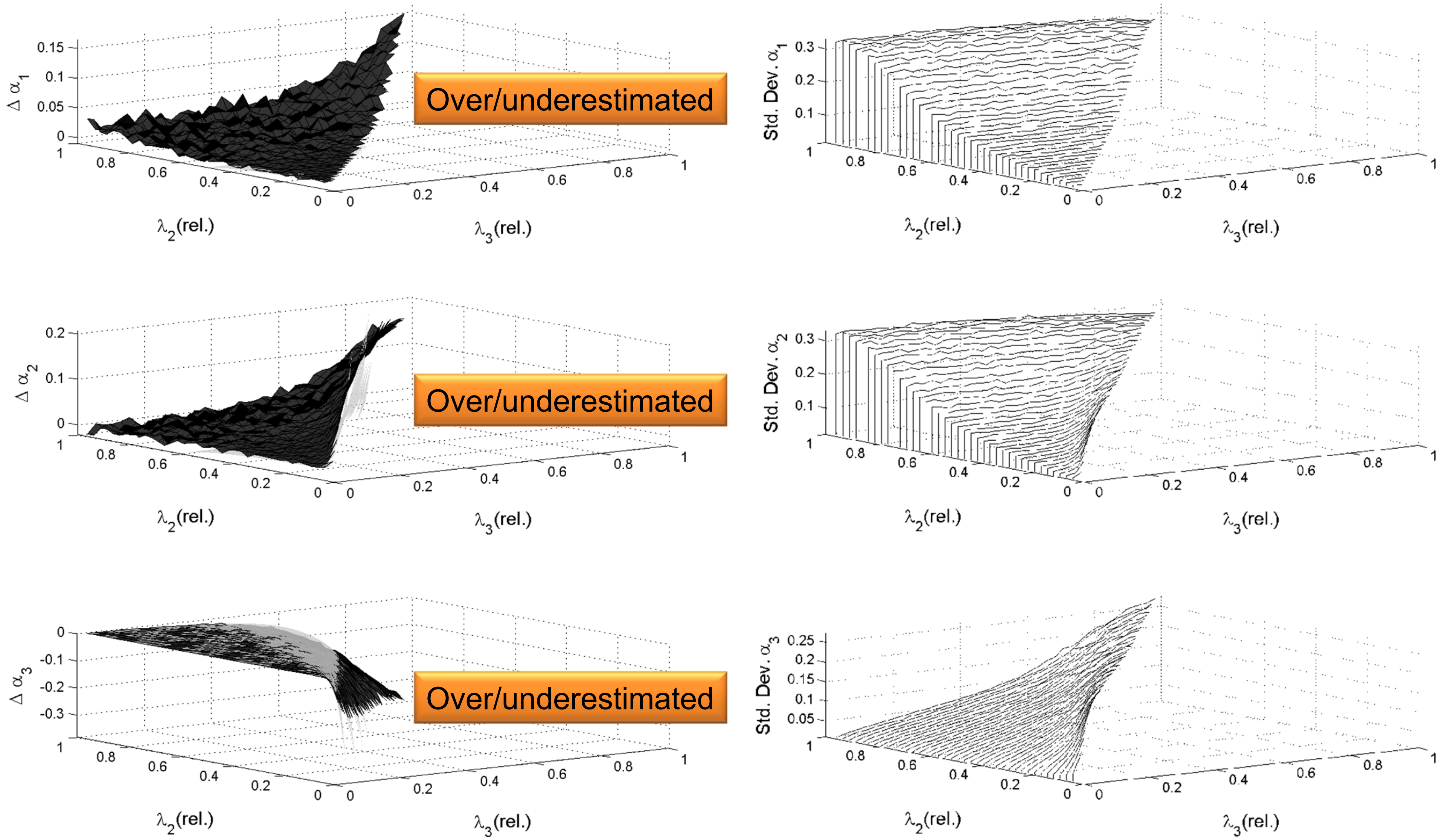
Analytical expressions

$$\alpha_1 = \frac{\pi}{4} rad$$

Evalutation in Terms of Simulated Data

Alpha angles bias | **Alpha angles Std. Dev.**

First / **Second** / **Third**



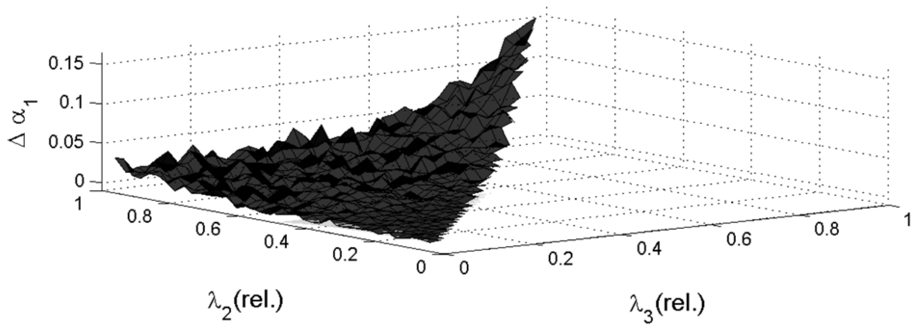
Simulated data

Analytical expressions

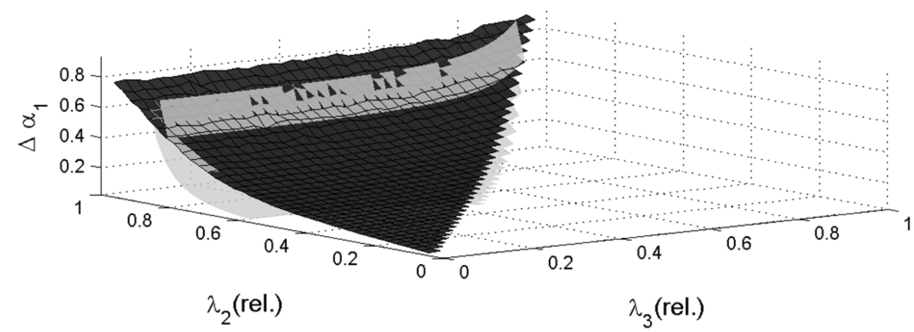
$$\alpha_1 = \frac{\pi}{4} rad$$

Evalutation in Terms of Simulated Data

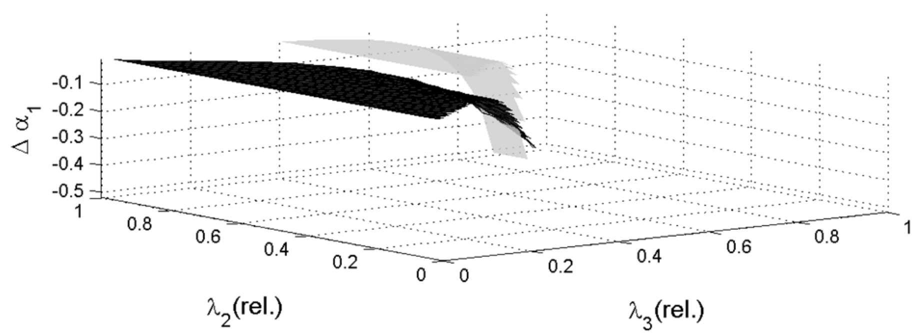
Alpha angles bias



$$\alpha_1 = \frac{\pi}{4}\, rad$$



$$\alpha_1 = 0\, rad$$



$$\alpha_1 = \frac{\pi}{2}\, rad$$

Over/underestimated

Simulated data
Analytical expressions

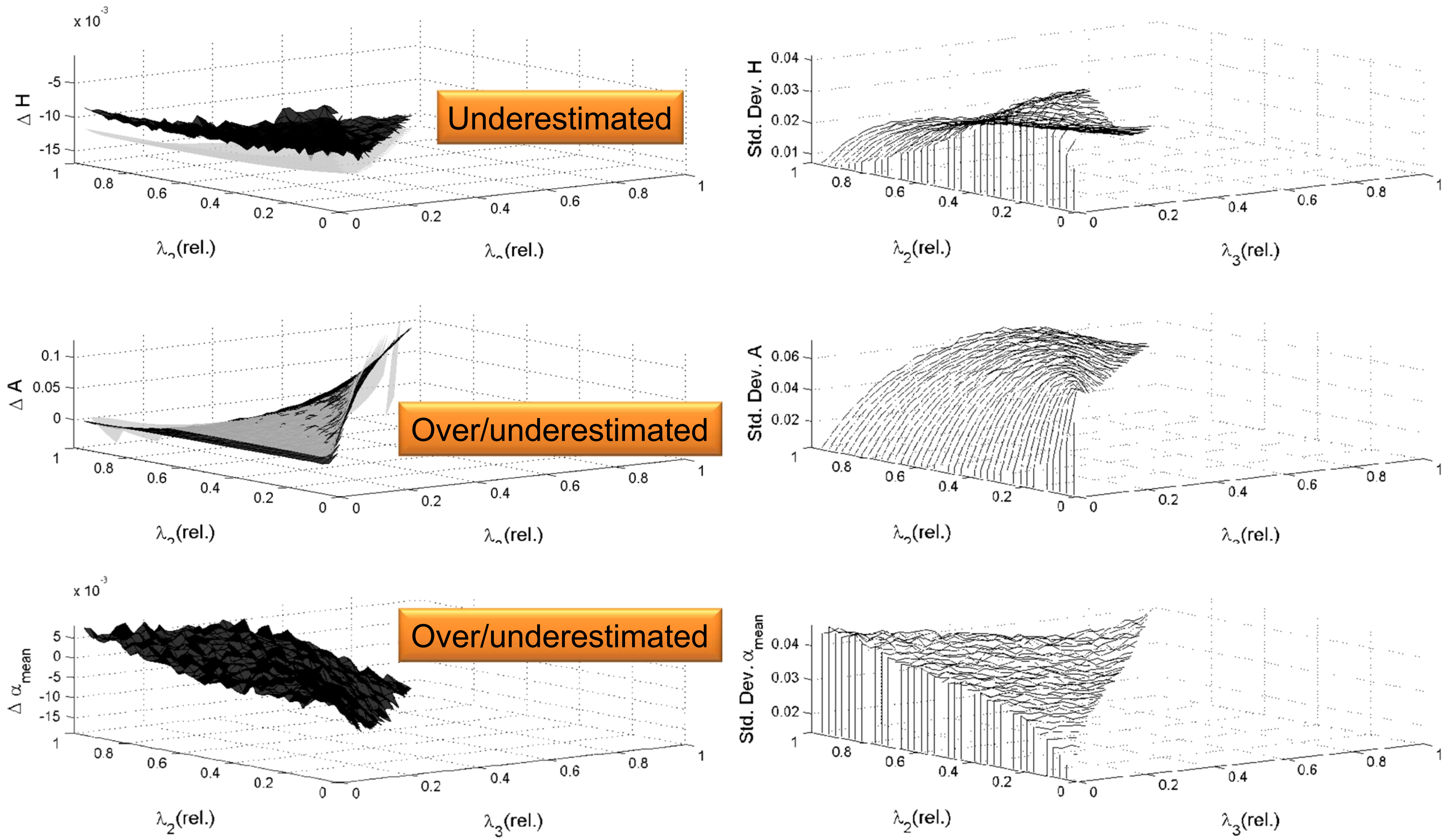
Evalutation in Terms of Simulated Data

Bias | **Std. Dev.**

Entropy

Anisotropy

Mean alpha



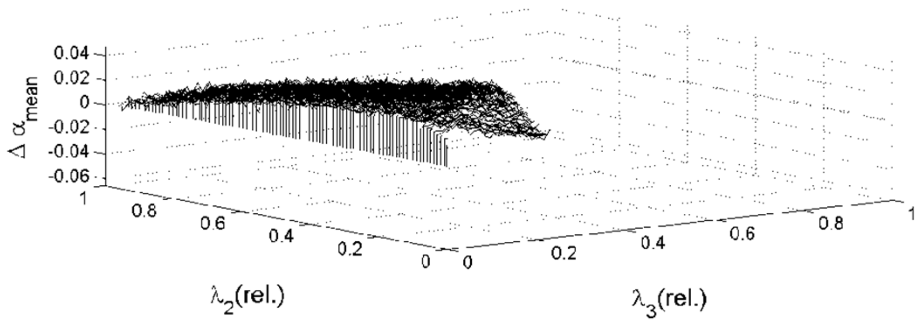
Simulated data

Analytical expressions

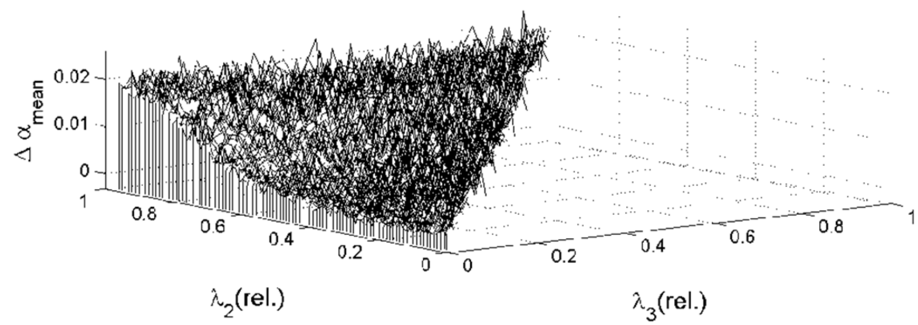
$$\alpha_1 = \frac{\pi}{4} rad$$

Evalutation in Terms of Simulated Data

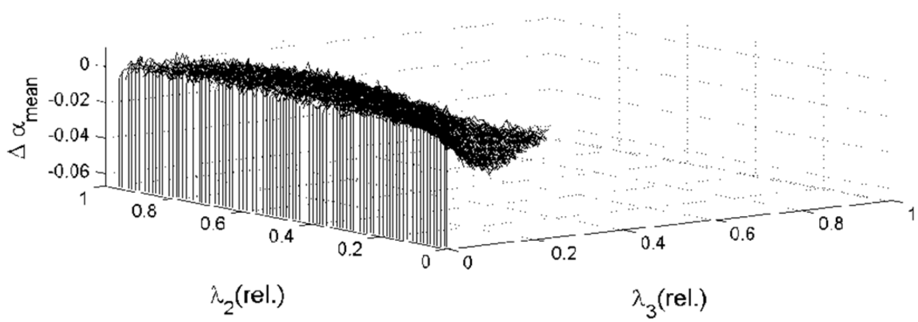
Mean Alpha angle bias



$$\alpha_1 = 0\ rad$$



$$\alpha_1 = 1\ rad$$



$$\alpha_1 = 2\ rad$$

Over/underestimated

Simulated data
Analytical expressions

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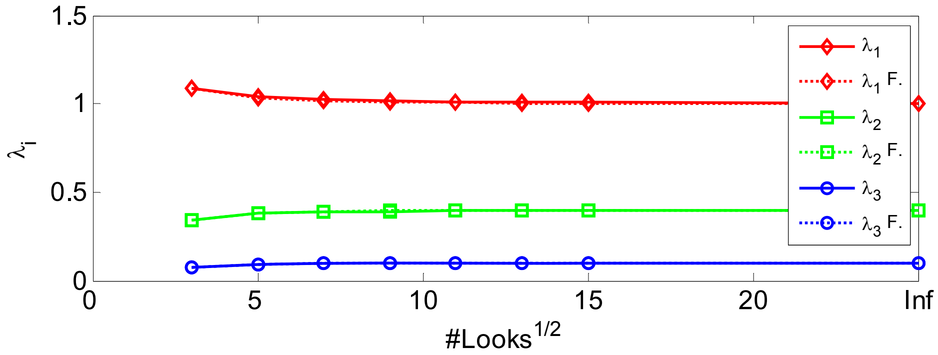
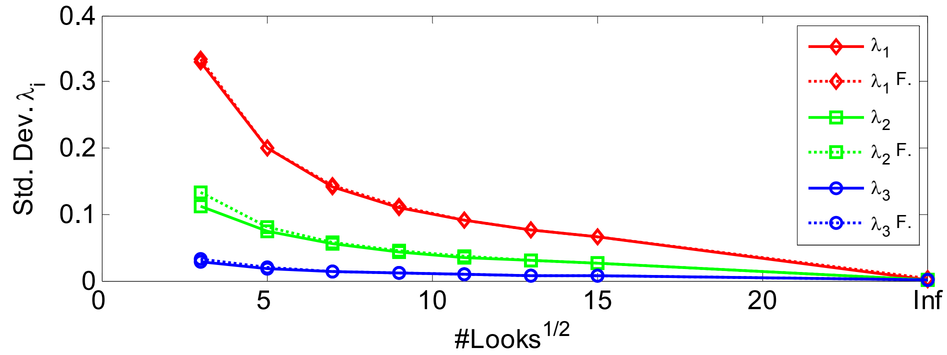
Evalutation in Terms of Simulated Data

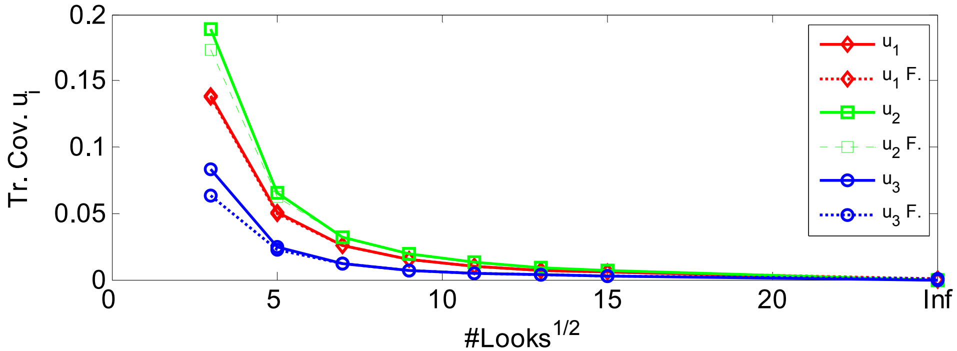
Analysis in terms of averaged samples/speckle filter

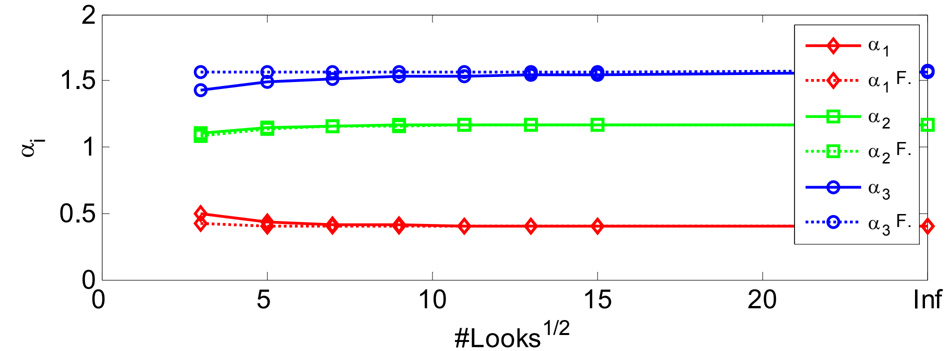
- Eigenvalues

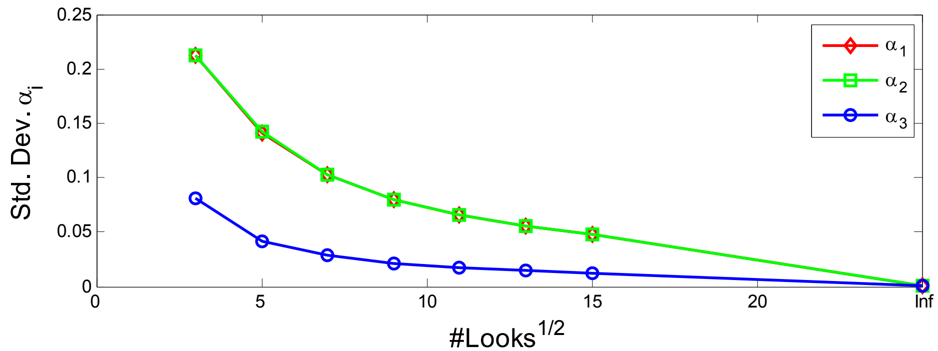




- Eigenvectors





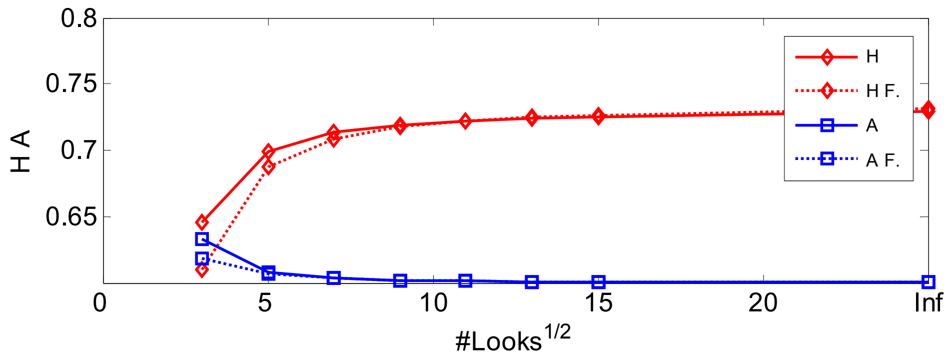


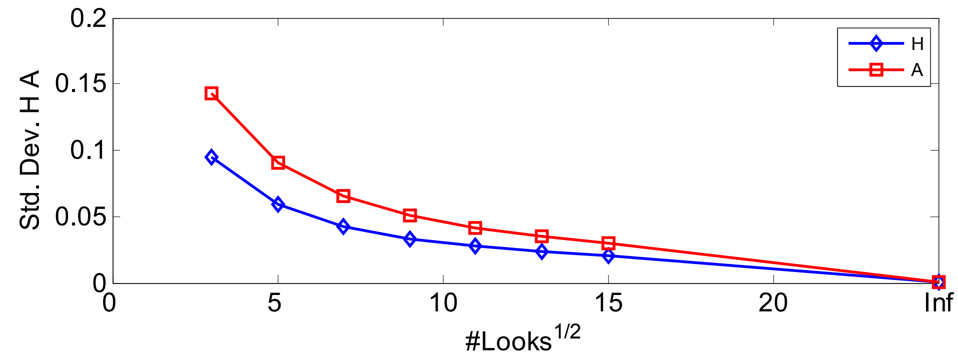
F. Analytical expression

Evalutation in Terms of Simulated Data

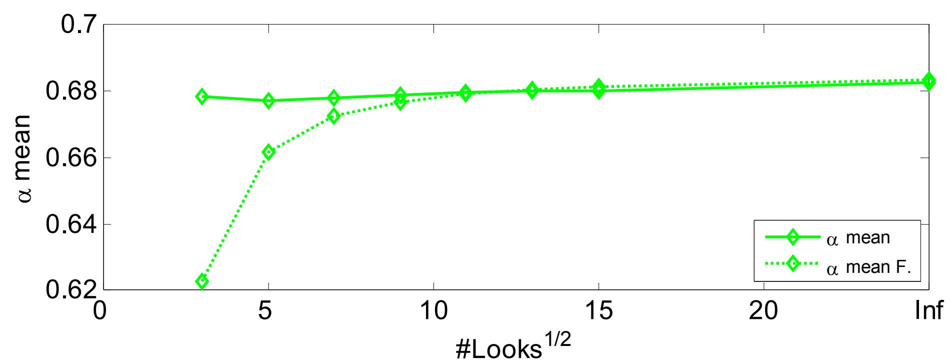
Analysis in terms of averaged samples/speckle filter

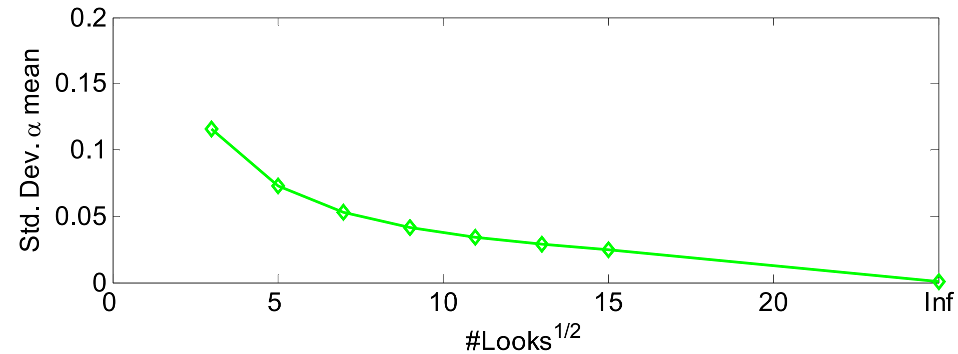
- Entropy and Anisotropy





- Mean Alpha angle

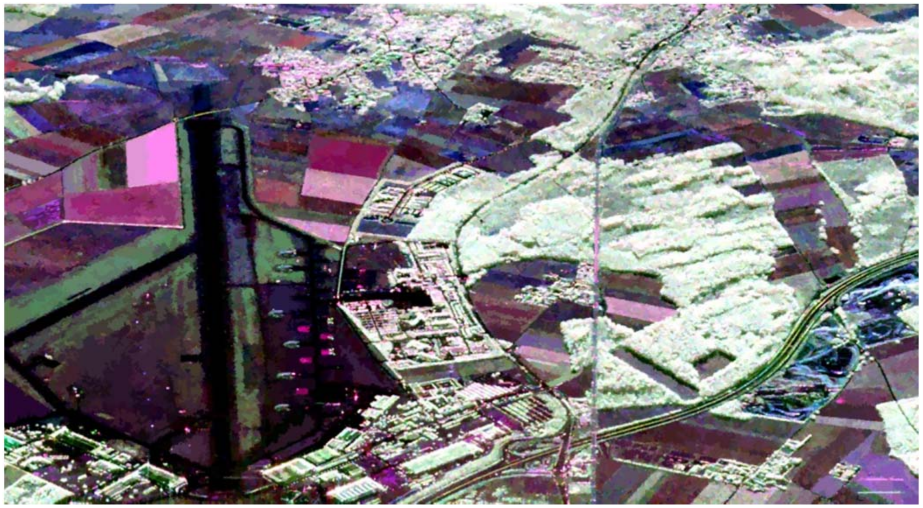




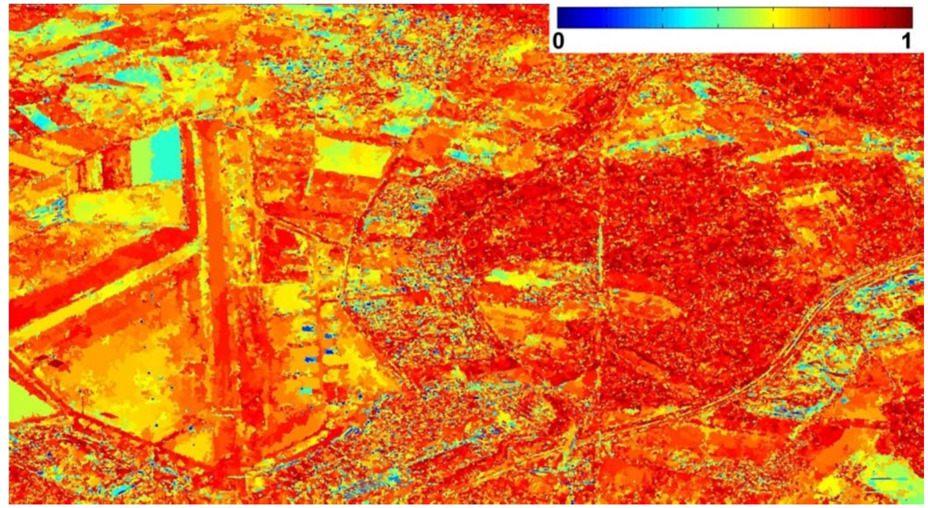
Evalutation in Terms of Real Data

Oberfapfenoffen L-band ESAR dataset

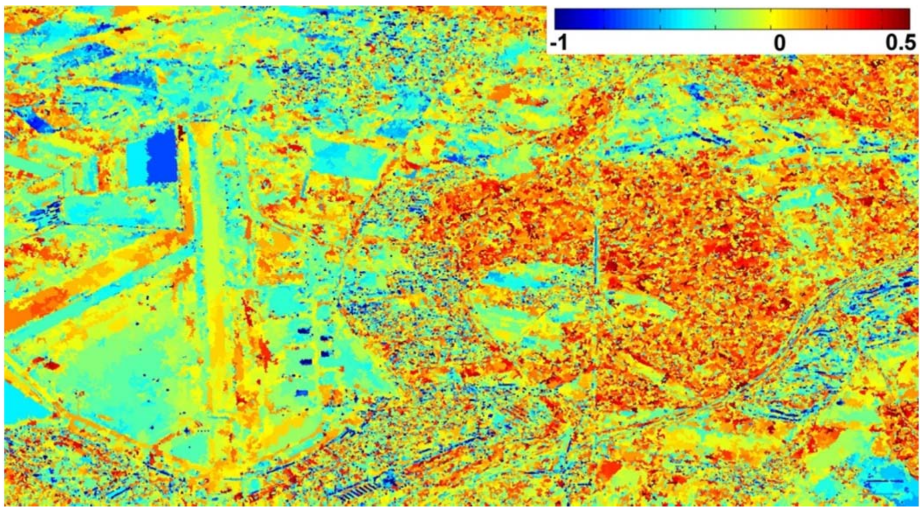
|HH+VV| |HV| |HH-VV|



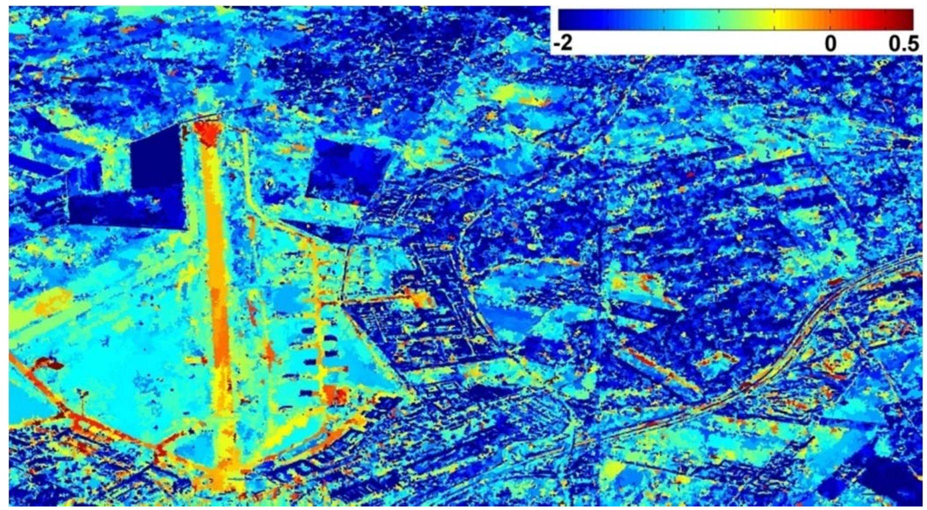
Entropy



Eigvalue Total Relative Bias



Eigvector Total Relative Bias



$$\Delta_{12}^{\lambda} = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \qquad \Delta_{13}^{\lambda} = \frac{\lambda_1 \lambda_3}{\lambda_1 - \lambda_3} \qquad \Delta_{23}^{\lambda} = \frac{\lambda_2 \lambda_3}{\lambda_2 - \lambda_3}$$

$$\Delta_{TRB}^{\lambda} = \frac{\Delta_{12}^{\lambda} + \Delta_{13}^{\lambda} + \Delta_{23}^{\lambda}}{\sum_{i=1}^{3} \lambda_i}$$

$$\Delta_{12}^{\mathbf{u}} = \frac{\lambda_1 \lambda_2}{\left(\lambda_1 - \lambda_2\right)^2} \qquad \Delta_{13}^{\mathbf{u}} = \frac{\lambda_1 \lambda_3}{\left(\lambda_1 - \lambda_3\right)^2} \qquad \Delta_{23}^{\mathbf{u}} = \frac{\lambda_2 \lambda_3}{\left(\lambda_2 - \lambda_3\right)^2}$$

$$\Delta_{TRB}^{\mathbf{u}} = \frac{\Delta_{12}^{\mathbf{u}} + \Delta_{13}^{\mathbf{u}} + \Delta_{23}^{\mathbf{u}}}{\sum_{i=1}^{3} \lambda_i}$$

Conclusions

Deeper and complete characterization of the speckle noise effects in the eigendecomposition of the covariance/coherency matrices:

- All the parameters are asymptotically non biased
- **Eigenvalues** over/underestimated
  - Variance proportional to λ_1^2
- **Entropy** underestimated
- **Anisotropy** over/underestimated
- **Eigenvectors** biased
  - Depend on true eigenvalues and eigenvectors
- **Alpha angles** over/underestimated
- **Mean Alpha angle** over/underestimated

$$\frac{1}{\lambda_i - \lambda_k}$$

$$\frac{1}{\left(\lambda_i - \lambda_k\right)^2} \quad \left|\mathbf{u}_i(1)\right|^2$$

Limitations of the perturbation analysis:

- Magnitude of the perturbation □
- Presence of very close eigenvalues
  - Eigenvalues with a multiplicity 2 or 3