SNR and Noise Variance Estimation in Polarimetric SAR Data

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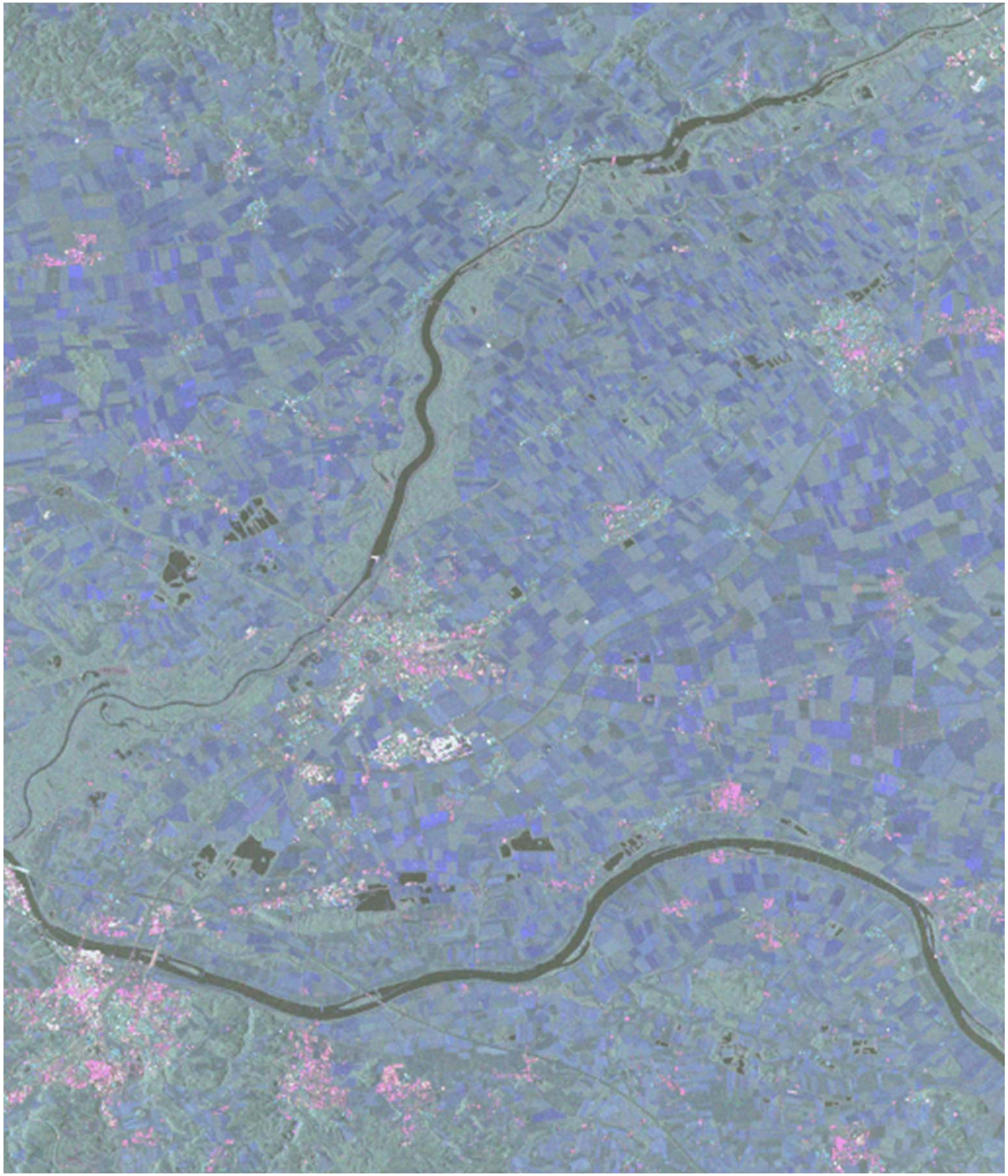
POLINSAR 2013
28 January 2013 – ESA-ESRIN



Pauli Color-Coded Image

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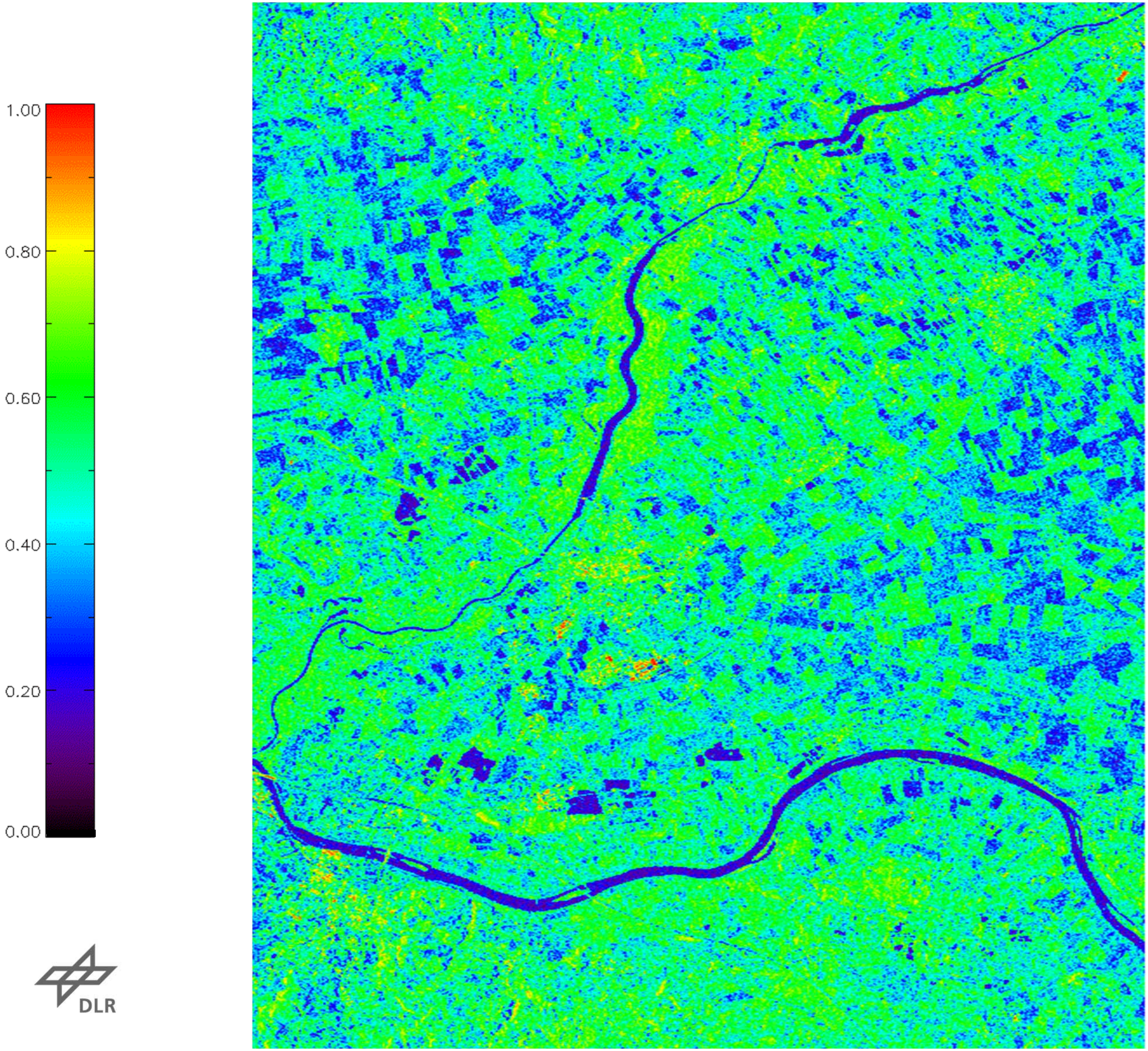


HV-VH Coherence

Coherency Matrix

Noisy Data

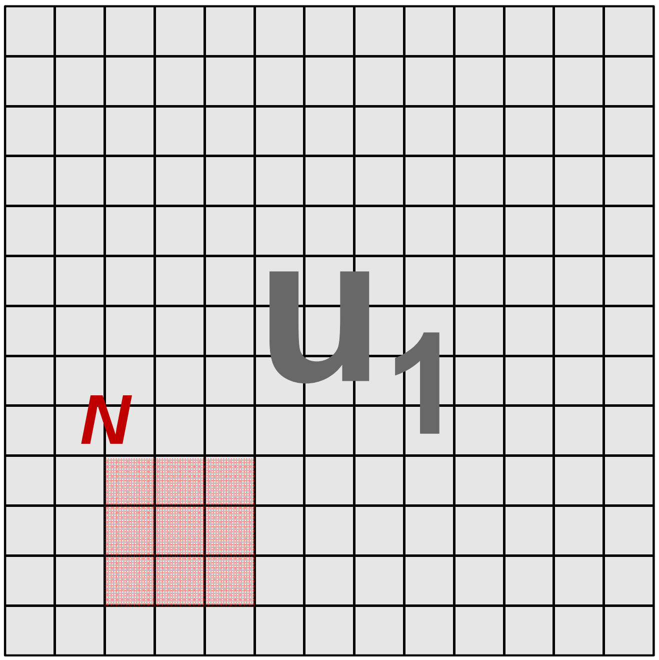
$$\mathbf{T}_n = \begin{bmatrix} T_{11} + \sigma^2 & T_{12} & T_{13} \\ T_{21} & T_{22} + \sigma^2 & T_{23} \\ T_{31} & T_{32} & T_{33} + \sigma^2 \end{bmatrix}$$

Noise-free Data

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

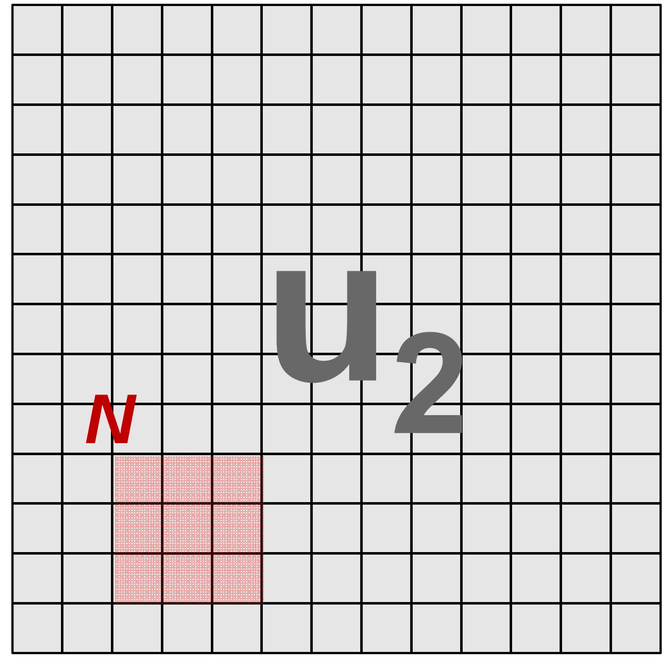
Model for Cross-Pol Data

HV



$$u_1[i] = s[i] + w_1[i]$$

VH



$$u_2[i] = s[i] + w_2[i]$$

$$\sigma^2 = E\left\{ |w_1|^2 \right\} = E\left\{ |w_2|^2 \right\}$$

$$SNR = \frac{E\left\{ |s|^2 \right\}}{E\left\{ |w_1|^2 \right\}} = \frac{E\left\{ |s|^2 \right\}}{E\left\{ |w_2|^2 \right\}}$$

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Cramér–Rao Lower Bound (CRLB)

$$u_1[i] = s[i] + w_1[i], i = 0..N-1$$
$$u_2[i] = s[i] + w_2[i], i = 0..N-1$$

$$\mathbf{x} = [u_1[0] \;\; u_1[1] \;\; \cdots \;\; u_1[N-1] \;\; u_2[0] \;\; u_2[1] \;\; \cdots \;\; u_2[N-1]]^T$$

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\pi^{2N} \det(\mathbf{C}_{\mathbf{x}})} \exp\left\{-\mathbf{x}^H \mathbf{C}_{\mathbf{x}}^{-1} \mathbf{x}\right\}$$

$$\ln[p_{\mathbf{x}}(\mathbf{x})] = -2N \ln(\pi) - 2N \ln(\sigma^2) - N \ln(2SNR+1) - \frac{1}{\sigma^2}\left[\frac{SNR+1}{2SNR+1}\sum_{i=0}^{2N-1}|x_i|^2 - \frac{SNR}{2SNR+1}\sum_{i=0}^{N-1} 2\,\mathrm{Re}\left(x_i^* x_{i+N}\right)\right]$$

$$\mathbf{J}(SNR,\sigma^2) = \begin{bmatrix} -E\left\{\frac{\partial^2 \ln[p_{\mathbf{x}}(\mathbf{x})]}{\partial SNR^2}\right\} & -E\left\{\frac{\partial^2 \ln[p_{\mathbf{x}}(\mathbf{x})]}{\partial SNR\,\partial(\sigma^2)}\right\} \\ -E\left\{\frac{\partial^2 \ln[p_{\mathbf{x}}(\mathbf{x})]}{\partial(\sigma^2)\,\partial SNR}\right\} & -E\left\{\frac{\partial^2 \ln[p_{\mathbf{x}}(\mathbf{x})]}{\partial(\sigma^2)^2}\right\} \end{bmatrix} = \begin{bmatrix} \frac{4N}{(2SNR+1)^2} & \frac{2N}{\sigma^2(2SNR+1)} \\ \frac{2N}{\sigma^2(2SNR+1)} & \frac{2N}{\sigma^4} \end{bmatrix}$$

$$\mathbf{J}^{-1}(SNR,\sigma^2) = \begin{bmatrix} \frac{(2SNR+1)^2}{2N} & -\frac{\sigma^2(2SNR+1)}{2N} \\ -\frac{\sigma^2(2SNR+1)}{2N} & \frac{\sigma^4}{N} \end{bmatrix}$$

$$\mathrm{var}\left\{\hat{SNR}\right\} \geq \frac{(2SNR+1)^2}{2N} \qquad \mathrm{var}\left\{\hat{\sigma}^2\right\} \geq \frac{\sigma^4}{N}$$

DLR

Cramér–Rao Lower Bound (CRLB)

$$u_1[i] = s[i] + w_1[i], i = 0..N-1$$
$$u_2[i] = s[i] + w_2[i], i = 0..N-1$$

$$\mathbf{x} = [u_1[0] \;\; u_1[1] \;\; \cdots \;\; u_1[N-1] \;\; u_2[0] \;\; u_2[1] \;\; \cdots \;\; u_2[N-1]]^T$$

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\pi^{2N} \det(\mathbf{C}_{\mathbf{x}})} \exp\{-\mathbf{x}^H \mathbf{C}_{\mathbf{x}}^{-1} \mathbf{x}\}$$

$$\ln[p_{\mathbf{x}}(\mathbf{x})] = -2N \ln(\pi) - 2N \ln(\sigma^2) - N \ln(2SNR+1) - \frac{1}{\sigma^2}\left[\frac{SNR+1}{2SNR+1}\sum_{i=0}^{2N-1}|x_i|^2 - \frac{SNR}{2SNR+1}\sum_{i=0}^{N-1} 2\,\mathrm{Re}(x_i^* x_{i+N})\right]$$

Hp: σ^2 is known

$$\mathbf{J}(SNR) = -E\left\{\frac{\partial^2 \ln[p_{\mathbf{x}}(\mathbf{x})]}{\partial SNR^2}\right\} = \frac{4N}{(2SNR+1)^2}$$

$$\mathbf{J}^{-1}(SNR) = \frac{(2SNR+1)^2}{4N}$$

Joint estimation of SNR and σ^2

$$\mathrm{var}\{\hat{SNR}\} \geq \frac{(2SNR+1)^2}{2N} \qquad \mathrm{var}\{\hat{\sigma}^2\} \geq \frac{\sigma^4}{N}$$

$$\mathrm{var}\{\hat{SNR}\} \geq \frac{(2SNR+1)^2}{4N}$$

Maximum Likelihood (ML) Estimation

$$u_1[i] = s[i] + w_1[i], i = 0..N-1$$
$$u_2[i] = s[i] + w_2[i], i = 0..N-1$$

$$\mathbf{x} = [u_1[0] \;\; u_1[1] \;\; \cdots \;\; u_1[N-1] \;\; u_2[0] \;\; u_2[1] \;\; \cdots \;\; u_2[N-1]]^T$$

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{2N \det(\mathbf{C}_{\mathbf{x}})} \exp\left\{- \mathbf{x}^H \mathbf{C}_{\mathbf{x}}^{-1} \mathbf{x}\right\}$$

$$\ln[p_{\mathbf{x}}(\mathbf{x})] = -2N \ln(\pi) - 2N \ln(\sigma^2) - N \ln(2SNR+1) - \frac{1}{\sigma^2}\left[\frac{SNR+1}{2SNR+1}\sum_{i=0}^{2N-1}|x_i|^2 - \frac{SNR}{2SNR+1}\sum_{i=0}^{N-1} 2\,\mathrm{Re}\left(x_i^* x_{i+N}\right)\right]$$

$$\begin{cases} \dfrac{\partial \ln[p_{\mathbf{x}}(\mathbf{x})]}{\partial SNR} = 0 \\ \dfrac{\partial \ln[p_{\mathbf{x}}(\mathbf{x})]}{\partial(\sigma^2)} = 0 \end{cases}$$

$$S\hat{N}R_{ML} = \frac{2\sum_{i=0}^{N-1} \mathrm{Re}\left\{u_1^*[i]\,u_2[i]\right\}}{\sum_{i=0}^{N-1}|u_1[i]-u_2[i]|^2}$$

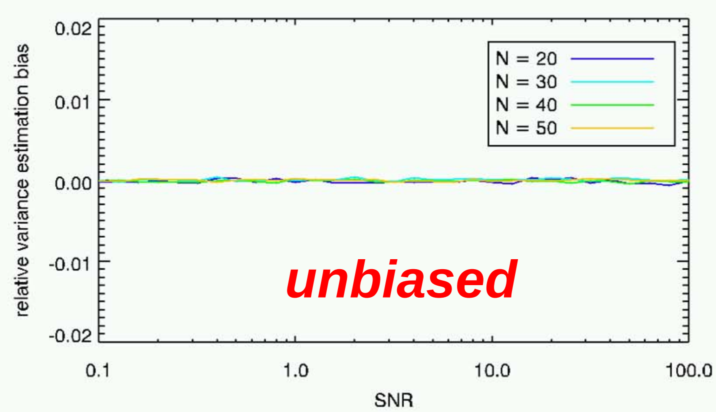
$$\hat{\sigma}^2_{ML} = \frac{\sum_{i=0}^{N-1}|u_1[i]-u_2[i]|^2}{2N}$$

Maximum Likelihood (ML) Estimation

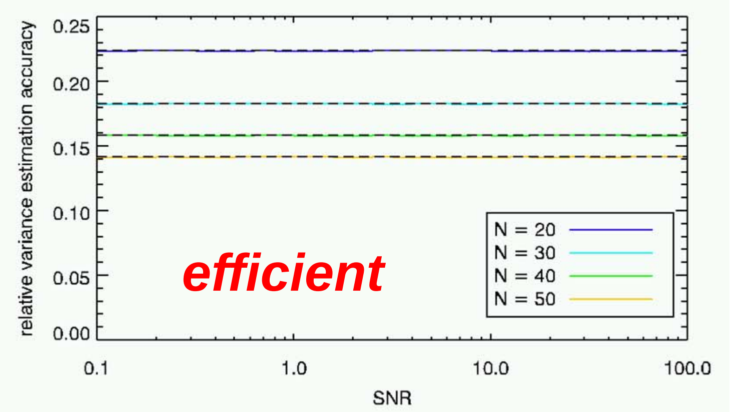
Noise Variance

$$\hat{\sigma}^2_{ML} = \frac{\sum_{i=0}^{N-1} \left|u_1[i]-u_2[i]\right|^2}{2N}$$

Relative Bias vs. SNR



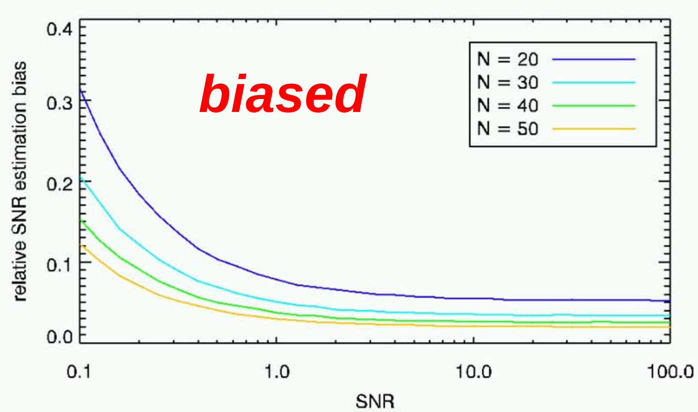
Relative Accuracy vs. SNR



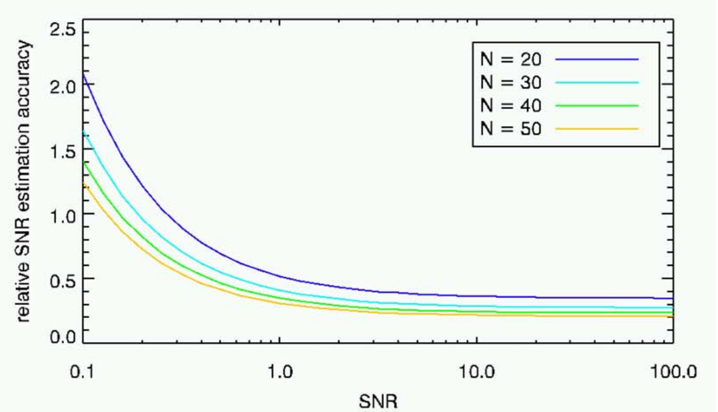
SNR

$$\hat{SNR}_{ML} = \frac{2\sum_{i=0}^{N-1} \mathrm{Re}\left\{u_1^*[i]\,u_2[i]\right\}}{\sum_{i=0}^{N-1} \left|u_1[i]-u_2[i]\right|^2}$$

Relative Bias vs. SNR



Relative Accuracy vs. SNR



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Maximum Likelihood (ML) Estimators

$$u_1[i] = s[i] + w_1[i], i = 0..N-1$$
$$u_2[i] = s[i] + w_2[i], i = 0..N-1$$

$$\mathbf{x} = [u_1[0] \;\; u_1[1] \;\; \cdots \;\; u_1[N-1] \;\; u_2[0] \;\; u_2[1] \;\; \cdots \;\; u_2[N-1]]^T$$

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{2N\,\det(\mathbf{C_x})} \exp\{-\mathbf{x}^H \mathbf{C}_{\mathbf{x}}^{-1} \mathbf{x}\}$$

$$\ln[p_{\mathbf{x}}(\mathbf{x})] = -2N\ln(\pi) - 2N\ln(\sigma^2) - N\ln(2SNR+1) - \frac{1}{\sigma^2}\left[\frac{SNR+1}{2SNR+1}\sum_{i=0}^{2N-1}|x_i|^2 - \frac{SNR}{2SNR+1}\sum_{i=0}^{N-1} 2\,\mathrm{Re}(x_i^* x_{i+N})\right]$$

Hp: σ^2 is known

$$\frac{\partial \ln[p_{\mathbf{x}}(\mathbf{x})]}{\partial SNR} = 0$$

$$\hat{SNR}_{ML}(\sigma^2) = \frac{\sum_{i=0}^{N-1} |u_1[i] + u_2[i]|^2}{4N\sigma^2} - \frac{1}{2}$$

DLR

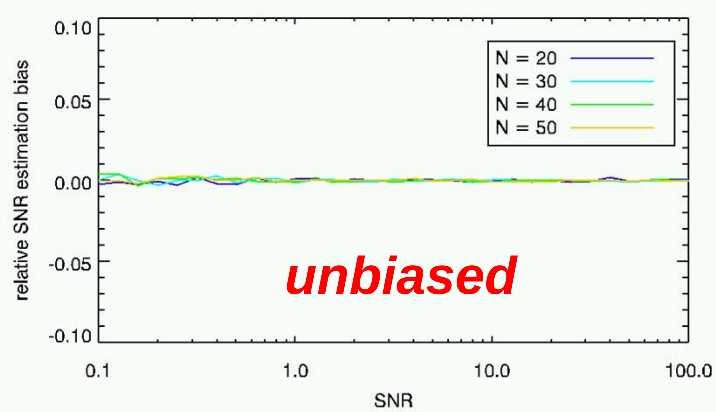
Maximum Likelihood (ML) Estimation

Hp: σ^2 is known

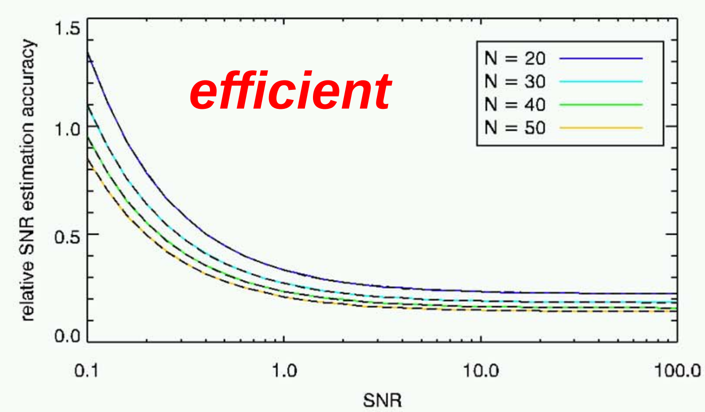
SNR

$$\hat{SNR}_{ML}\left(\sigma^2\right)=\frac{\sum_{i=0}^{N-1}\left|u_1[i]+u_2[i]\right|^2}{4N\sigma^2}-\frac{1}{2}$$

Relative Bias vs. SNR



Relative Accuracy vs. SNR



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Other Estimators

Eigenvalue-Based (EB) Noise Variance Estimator

$$\hat{\sigma}_{EB}^2 = \frac{1}{2N}\sum_{i=0}^{N-1}\left|u_1[i]\right|^2 + \frac{1}{2N}\sum_{i=0}^{N-1}\left|u_2[i]\right|^2 - \sqrt{\left(\frac{1}{2N}\sum_{i=0}^{N-1}\left|u_1[i]\right|^2 + \frac{1}{2N}\sum_{i=0}^{N-1}\left|u_2[i]\right|^2\right)^2 - \left(\frac{1}{N}\sum_{i=0}^{N-1}\left|u_1[i]\right|^2\right)\left(\frac{1}{N}\sum_{i=0}^{N-1}\left|u_2[i]\right|^2\right) + \left|\frac{1}{N}\sum_{i=0}^{N-1}u_1^*[i]u_2[i]\right|^2}$$

from S. Cloude, “Introduction to Pol-InSAR”,
Advanced Course on Radar Polarimetry, ESA-ESRIN, 2011

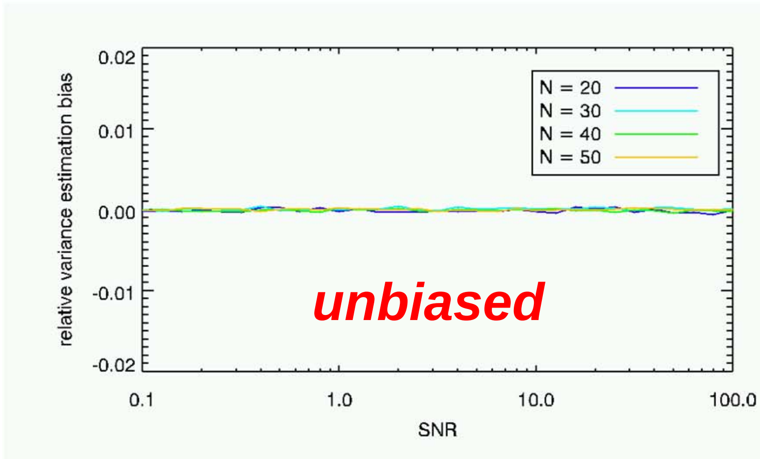
$$\mathrm{cov}\left(\begin{bmatrix} u_1 & u_2 \end{bmatrix}\right) = \begin{bmatrix} E\left\{u_1 u_1^*\right\} & E\left\{u_1 u_2^*\right\} \\ E\left\{u_1^* u_2\right\} & E\left\{u_2 u_2^*\right\} \end{bmatrix}$$

DLR

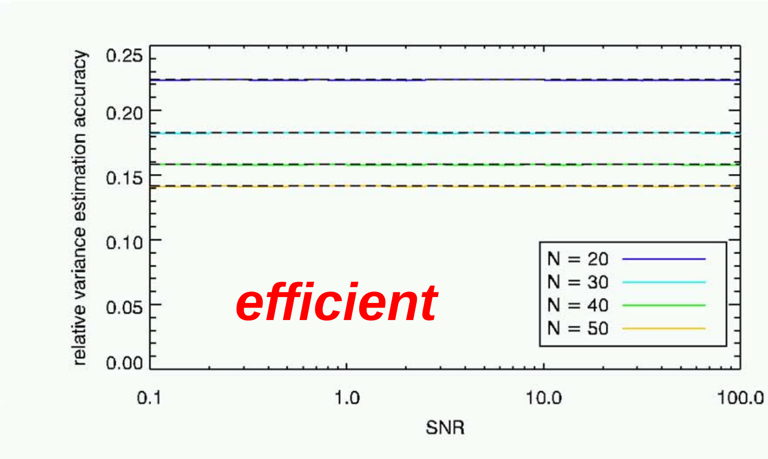
Other Estimators

Maximum Likelihood

Relative Bias vs. SNR

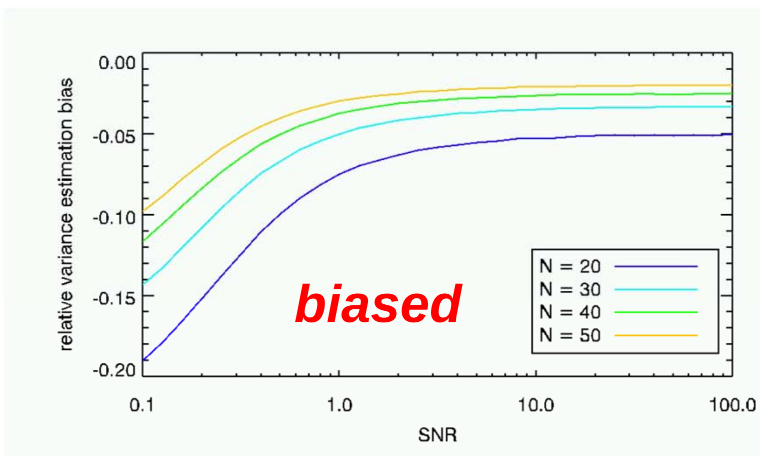


Relative Accuracy vs. SNR

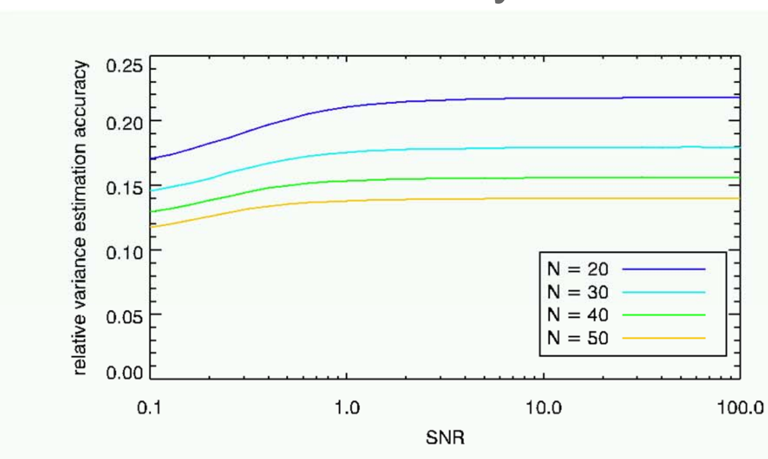


Eigenvalue-Based

Relative Bias vs. SNR

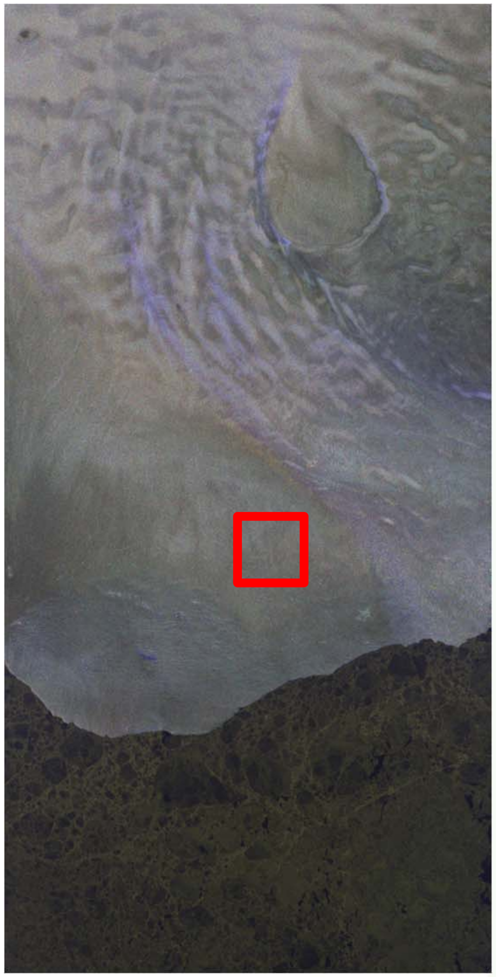


Relative Accuracy vs. SNR

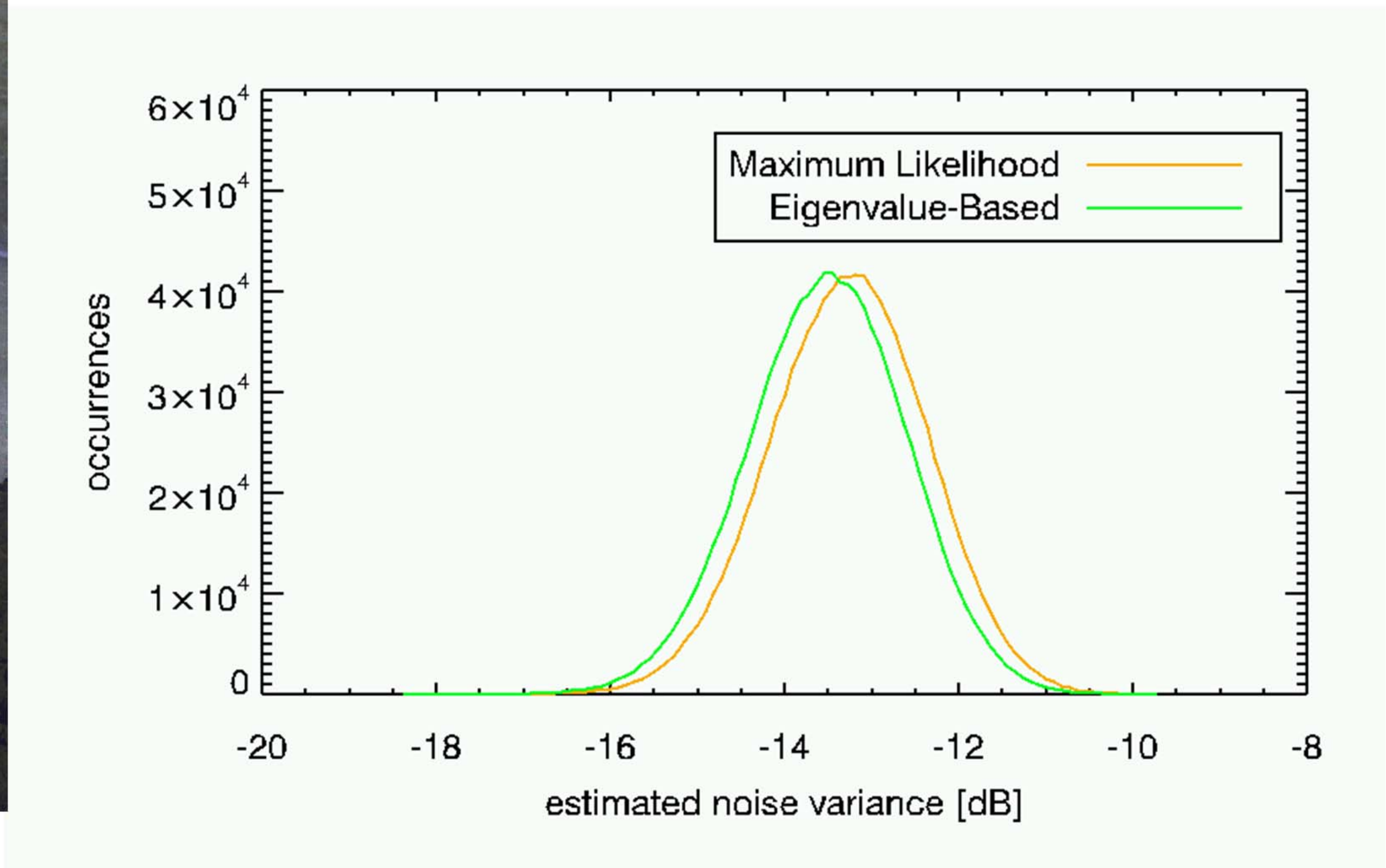


DLR

Other Estimators



Estimated σ^2 – Histogram



Other Estimators

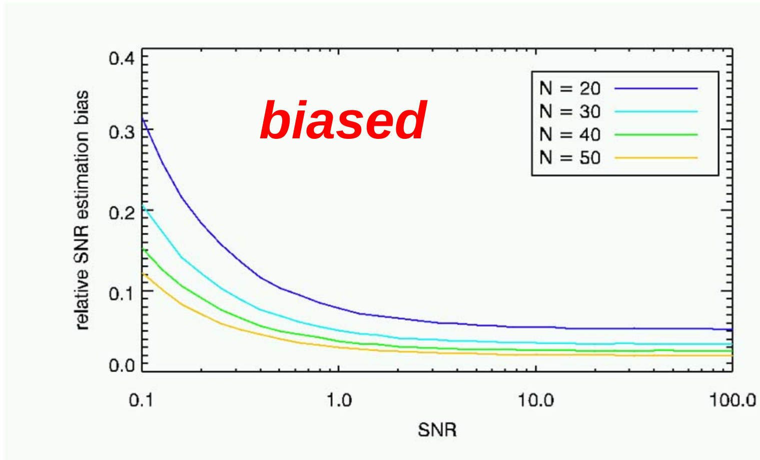
Coherence-Based (CB) SNR Estimator

$$S\hat{N}R_{CB} = \frac{\hat{\gamma}}{1-\hat{\gamma}} \qquad \hat{\gamma} = \frac{\left|\sum_{k=0}^{N-1} u_1[i]u_2^*[i]\right|}{\sqrt{\sum_{k=0}^{N-1}\left|u_1[i]\right|^2 \sum_{i=0}^{N-1}\left|u_2[i]\right|^2}}$$

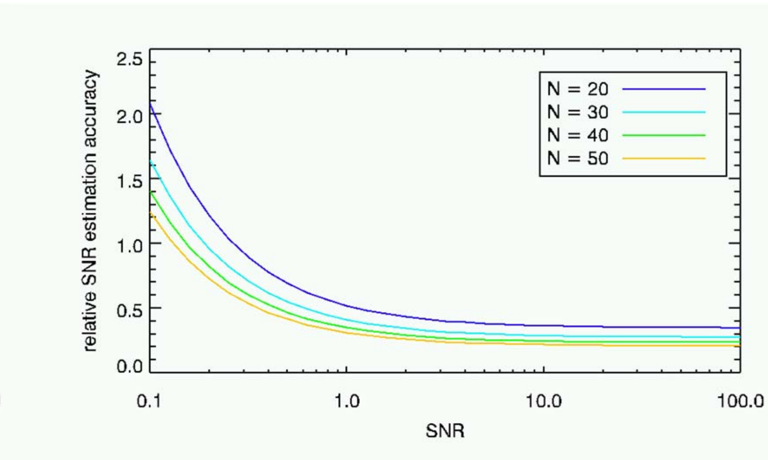
Other Estimators

Maximum Likelihood

Relative Bias vs. SNR

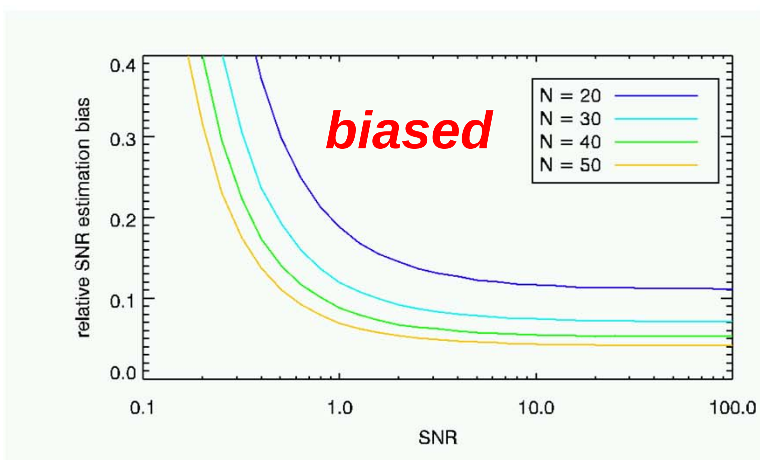


Relative Accuracy vs. SNR

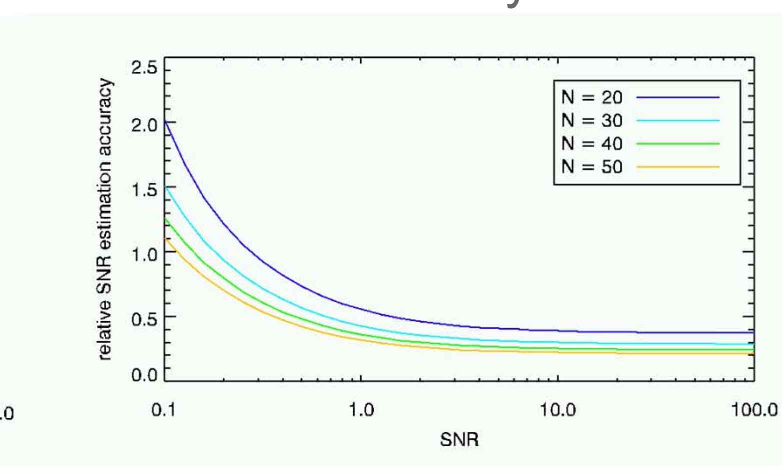


Coherence-Based

Relative Bias vs. SNR



Relative Accuracy vs. SNR



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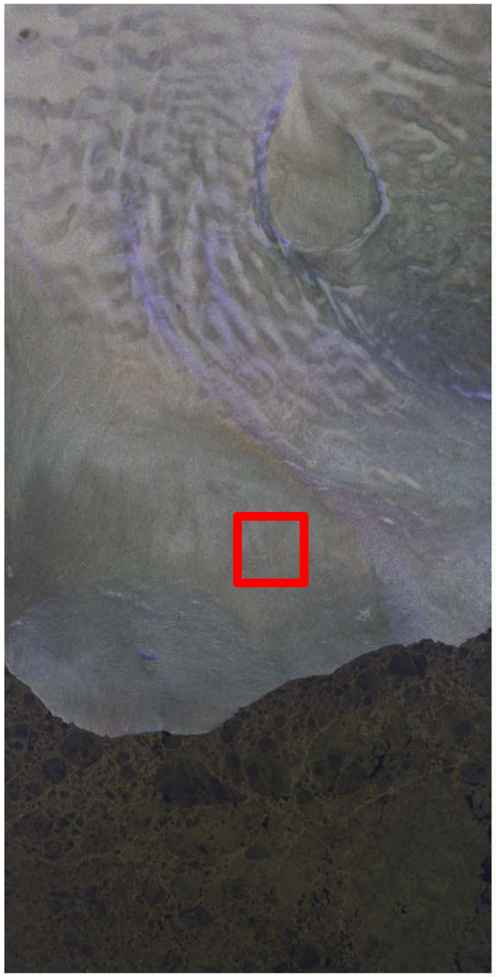
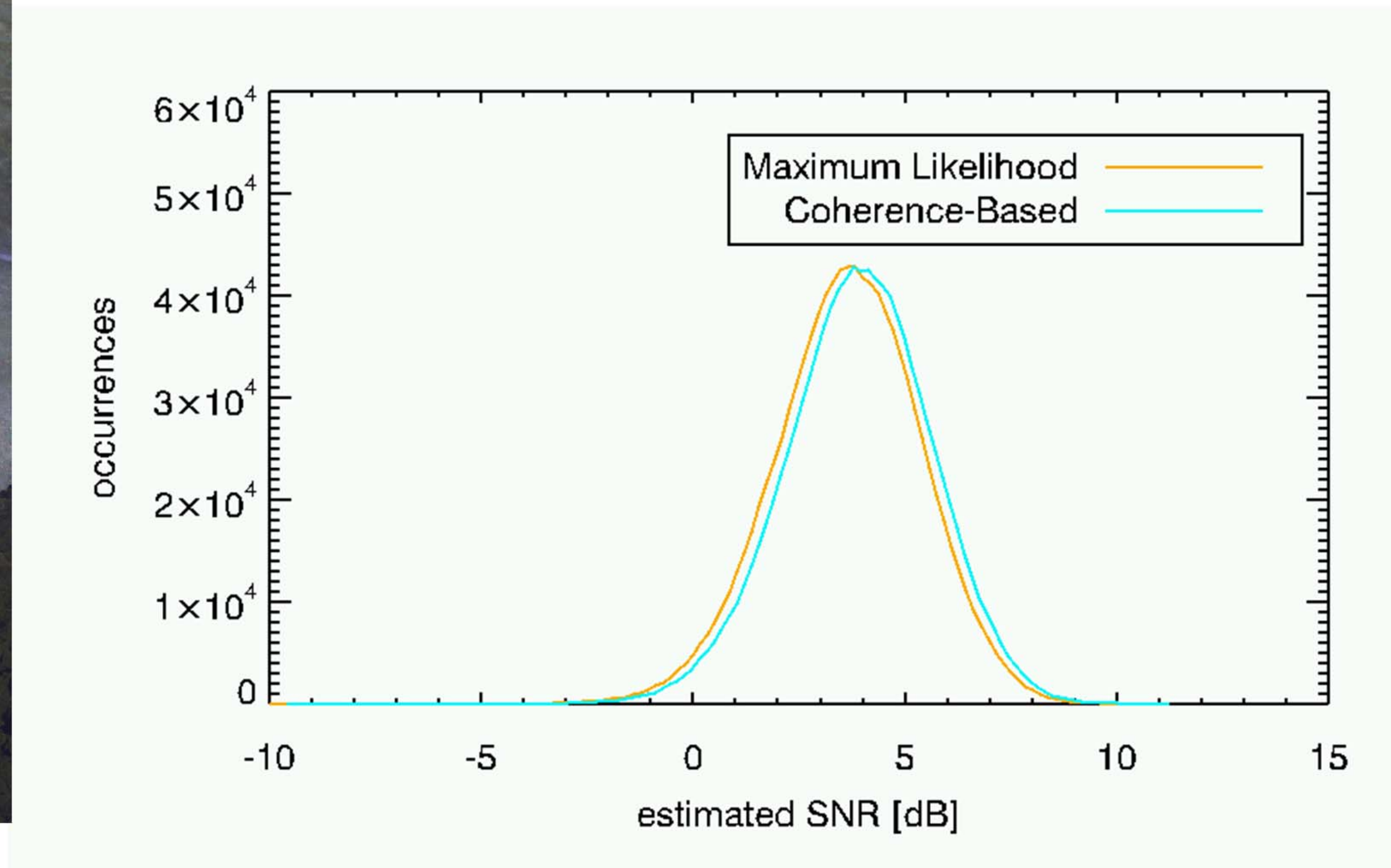
Other Estimators

Estimated SNR – Histogram



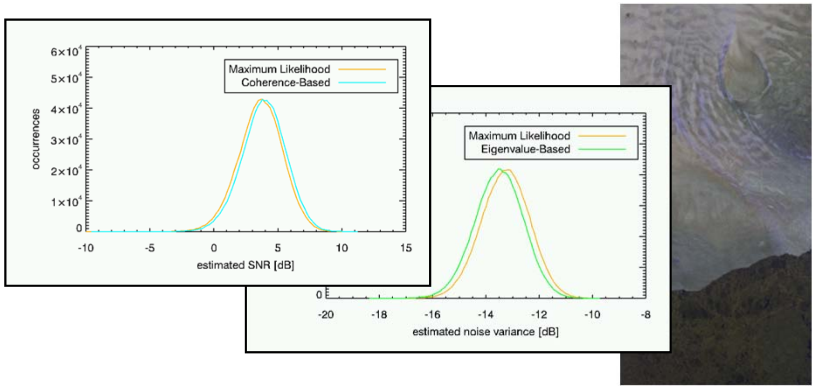


Hp: σ^2 is known

$$\mathrm{var}\{\hat{SNR}\} \geq \frac{(2SNR+1)^2}{4N}$$

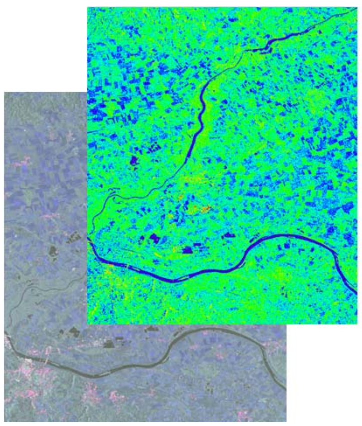
$$\mathrm{var}\{\hat{SNR}\} \geq \frac{(2SNR+1)^2}{2N} \qquad \mathrm{var}\{\hat{\sigma}^2\} \geq \frac{\sigma^4}{N}$$

Cramér-Rao Lower Bound



Coherence-Based

Eigenvalue-Based



$$\hat{SNR}_{ML} = \frac{2\sum_{i=0}^{N-1} \mathrm{Re}\{u_1^*[i]u_2[i]\}}{\sum_{i=0}^{N-1} |u_1[i]-u_2[i]|^2} \qquad \hat{\sigma}^2_{ML} = \frac{\sum_{i=0}^{N-1} |u_1[i]-u_2[i]|^2}{2N}$$

Maximum Likelihood

$$\hat{SNR}_{ML}(\sigma^2) = \frac{\sum_{i=0}^{N-1} |u_1[i]+u_2[i]|^2}{4N\sigma^2} - \frac{1}{2}$$

Hp: σ^2 is known

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Thank you!

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Data Covariance Matrix

$$u_1[i] = s[i] + w_1[i], i = 0..N-1$$
$$u_2[i] = s[i] + w_2[i], i = 0..N-1$$

$$\mathbf{x} = \left[u_1[0] \;\; u_1[1] \;\; \cdots \;\; u_1[N-1] \;\; u_2[0] \;\; u_2[1] \;\; \cdots \;\; u_2[N-1]\right]^T$$

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{2N \det(\mathbf{C_x})} \exp\left\{-\mathbf{x}^H \mathbf{C}_{\mathbf{x}}^{-1} \mathbf{x}\right\}$$

$$\mathbf{C_x} = ?$$

$$\mathbf{C}_{\mathbf{x}\,i,i} = E\left\{\left(s[i] + w_{1+floor(i/N)}[i]\right)\left(s[i] + w_{1+floor(i/N)}[i]\right)^*\right\} = A^2 + \sigma^2 = \sigma^2(SNR+1), i = 0..2N-1$$
$$\mathbf{C}_{\mathbf{x}\,i,i+N} = E\left\{\left(s[i] + w_1[i]\right)\left(s[i] + w_2[i]\right)^*\right\} = A^2 = \sigma^2 SNR, i = 0..N-1$$
$$\mathbf{C}_{\mathbf{x}\,i+N,i} = E\left\{\left(s[i] + w_2[i]\right)\left(s[i] + w_1[i]\right)^*\right\} = A^2 = \sigma^2 SNR, i = 0..N-1$$
$$\mathbf{C}_{\mathbf{x}\,i+N,i} = 0, \text{ for all the other elements of the matrix}$$

$$\mathbf{C_x} = \begin{bmatrix} \sigma^2(SNR+1)\mathbf{I_N} & \sigma^2 SNR\,\mathbf{I_N} \\ \sigma^2 SNR\,\mathbf{I_N} & \sigma^2(SNR+1)\mathbf{I_N} \end{bmatrix}$$

DLR

Data Covariance Matrix

$$u_1[i] = s[i] + w_1[i], i = 0..N-1$$
$$u_2[i] = s[i] + w_2[i], i = 0..N-1$$

$$\mathbf{x} = [u_1[0] \;\; u_1[1] \;\; \cdots \;\; u_1[N-1] \;\; u_2[0] \;\; u_2[1] \;\; \cdots \;\; u_2[N-1]]^T$$

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{2N \det(\mathbf{C_x})} \exp\left\{-\mathbf{x}^H \mathbf{C_x^{-1}} \mathbf{x}\right\}$$

$$\mathbf{C_x} = \begin{bmatrix} \sigma^2 (SNR+1)\mathbf{I_N} & \sigma^2 SNR\,\mathbf{I_N} \\ \sigma^2 SNR\,\mathbf{I_N} & \sigma^2 (SNR+1)\mathbf{I_N} \end{bmatrix}$$

$$\det(\mathbf{C_x}) = \sigma^{4N} (2SNR+1)^N$$

$$\mathbf{C_x^{-1}} = \frac{1}{\sigma^2} \begin{bmatrix} \frac{SNR+1}{2SNR+1}\mathbf{I_N} & -\frac{SNR}{2SNR+1}\mathbf{I_N} \\ -\frac{SNR}{2SNR+1}\mathbf{I_N} & \frac{SNR+1}{2SNR+1}\mathbf{I_N} \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

$$\det(\mathbf{M}) = \det(\mathbf{AD} - \mathbf{BC})$$

$$\mathbf{M}^{-1} = \begin{bmatrix} (\mathbf{A} - \mathbf{BD}^{-1}\mathbf{C})^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{CA}^{-1}\mathbf{B})^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{BD}^{-1}\mathbf{C})^{-1} & (\mathbf{D} - \mathbf{CA}^{-1}\mathbf{B})^{-1} \end{bmatrix}$$

DLR