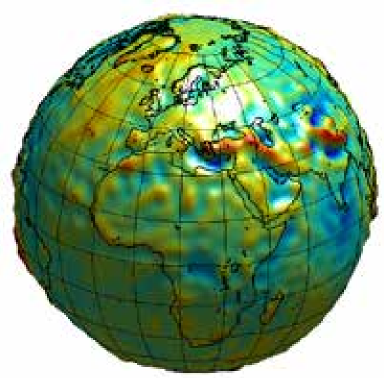
Lecture Two

Signal Processing on a Sphere

Three Lectures:

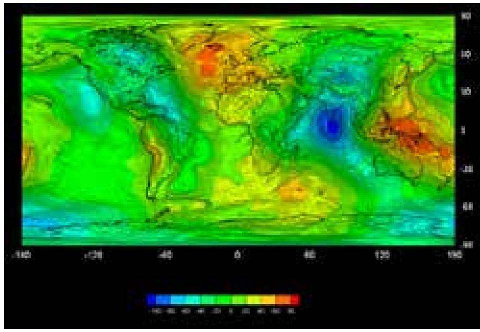
One	ESA explorer mission GOCE: earth gravity from space
Two	Signal processing on a sphere
Three	Gravity and earth sciences

functions on a sphere



Signal processing of functions on a sphere:

- integration, differentiation…
- spectral analysis
- filtering
- least-squares adjustment
- …



a matter of convenience:

- A Cartesian coordinate system in the earth‘s center

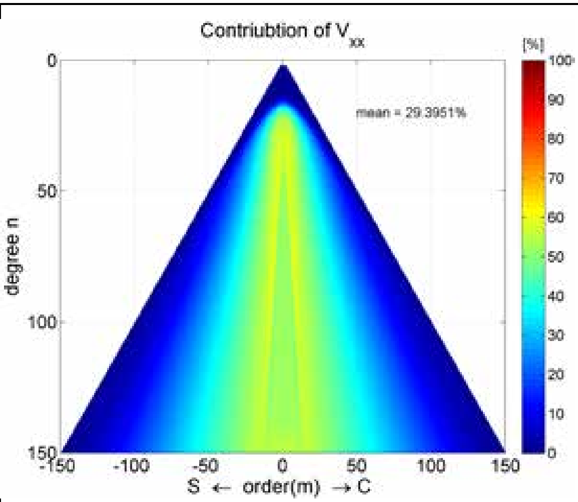
  {z-axis towards north pole, x-axis in Greenwich meridian plane, right-handed}

- Spherical coordinates: co-latitude θ (theta), (or latitude φ) and longitude λ (lambda)

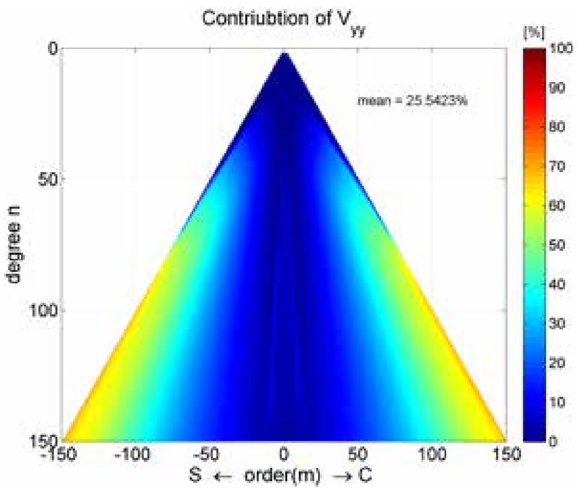
- Radius of sphere = 1 (unit sphere) or R

$$x = \sin\theta \cos\lambda$$
$$y = \sin\theta \sin\lambda$$
$$z = \cos\theta$$

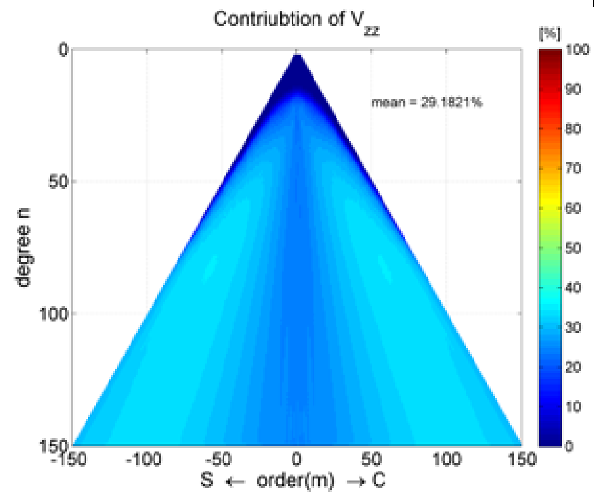
example: GOCE analysis of contributions



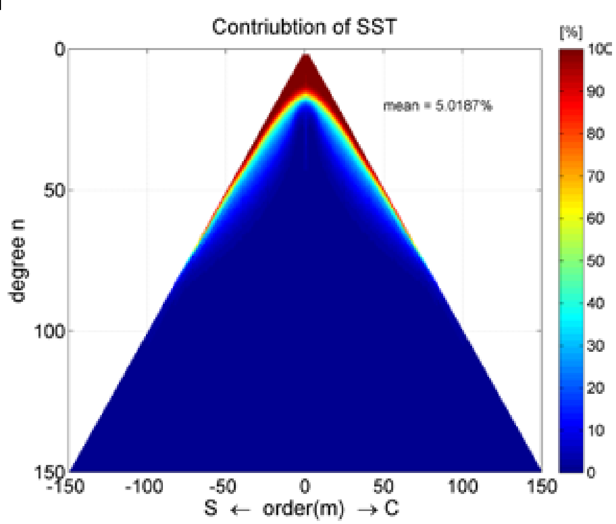
V_{xx} 29.4%



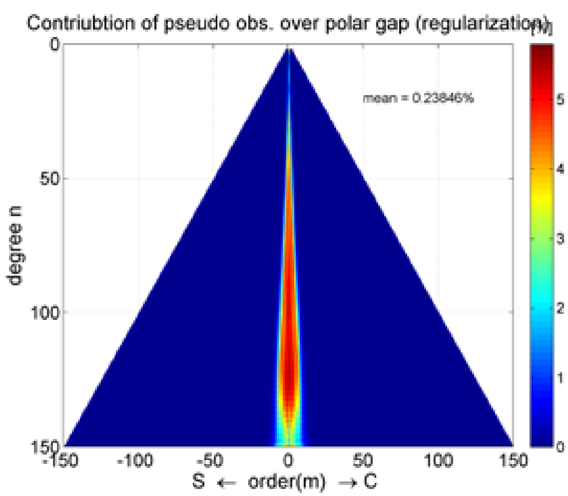
V_{yy} 25.5%



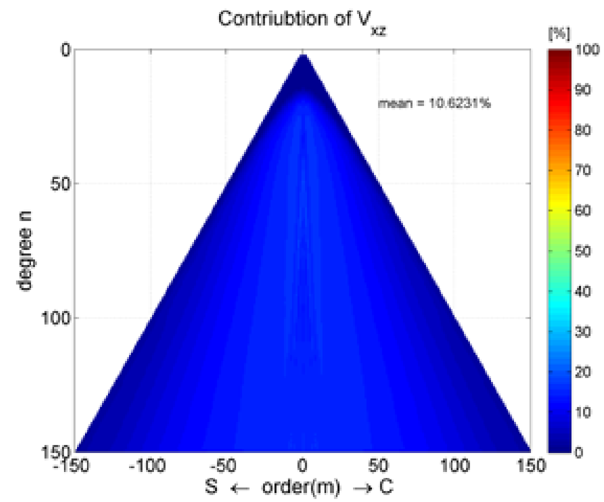
V_{zz} 29.2%



SST(GPS) 5.0%



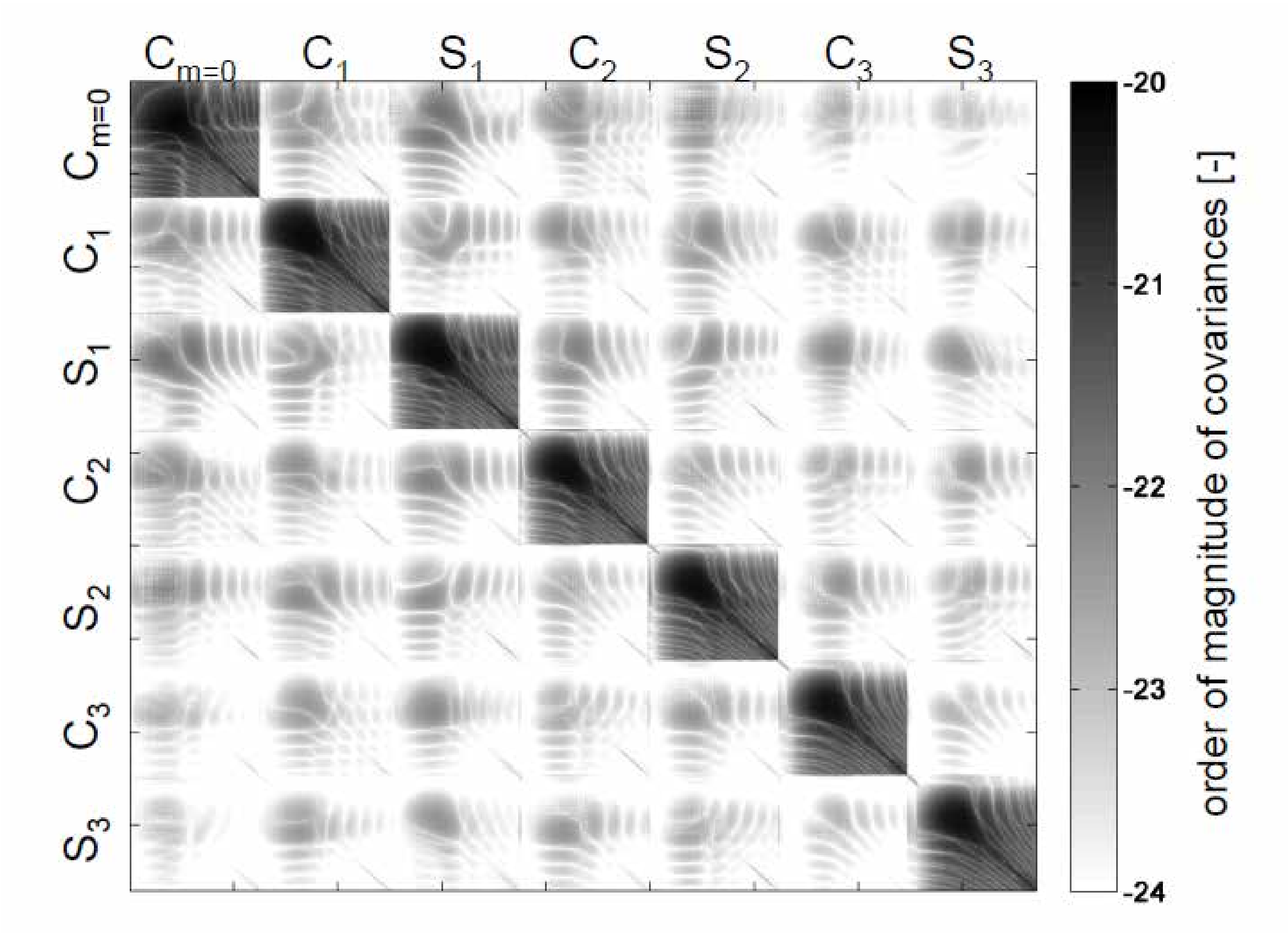
regularization 0.2%



V_{xz} 10.6%

Weiyong Yi, 2011

example: error variance-covariance propagation



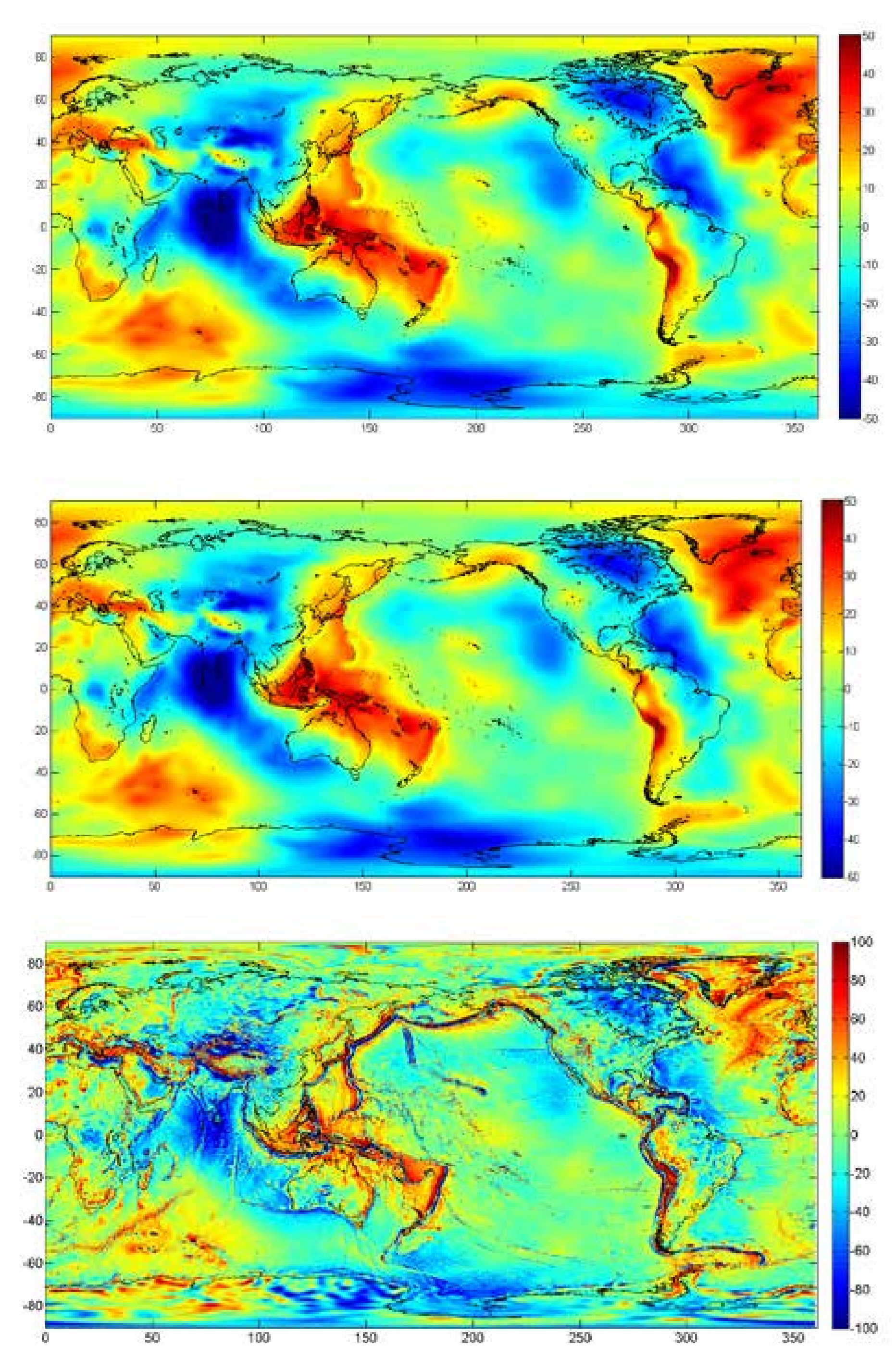
$T_r = \delta V_r$

example: gravity & attenuation with altitude

h = 400km

h = 250km

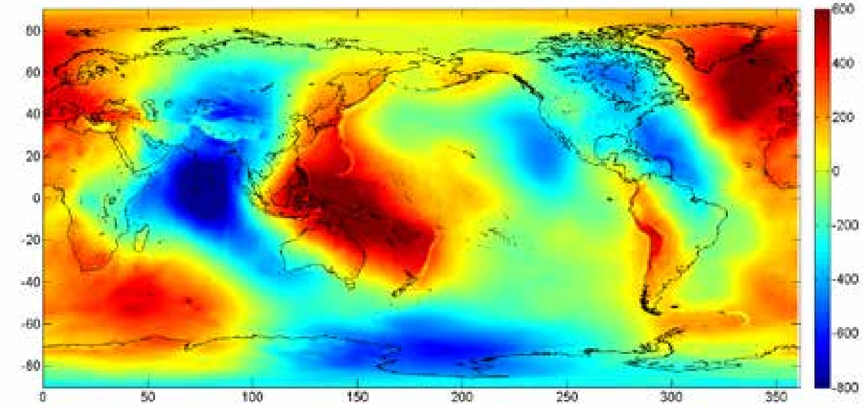
h = 0km



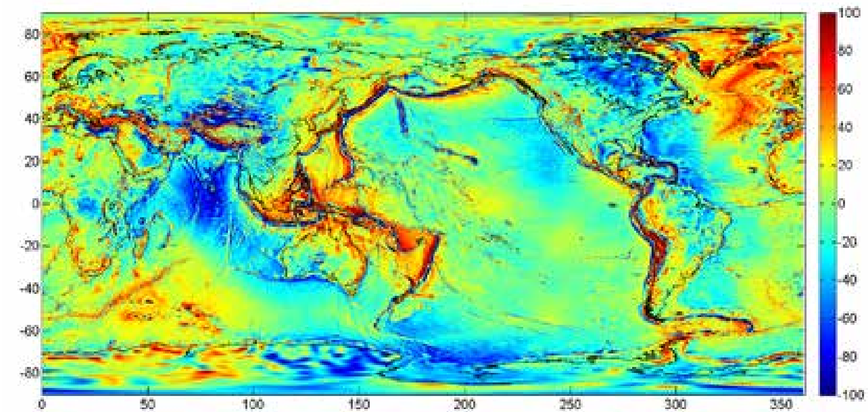
h = 0km

example:
potential,
first derivative,
second derivative

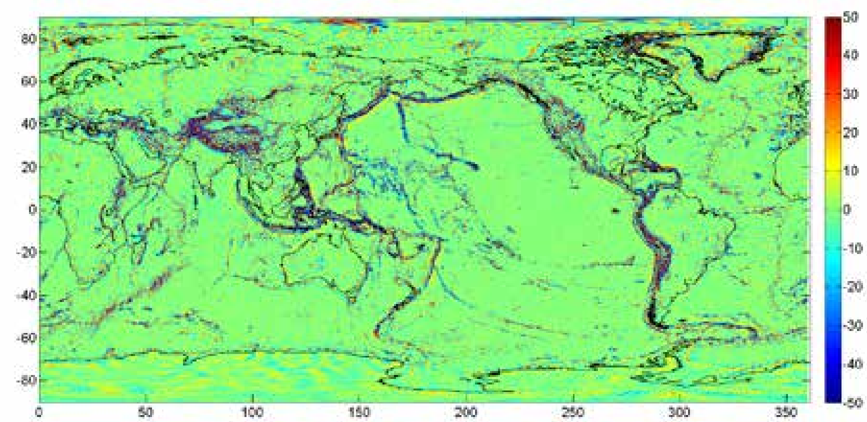
$T = \delta V$



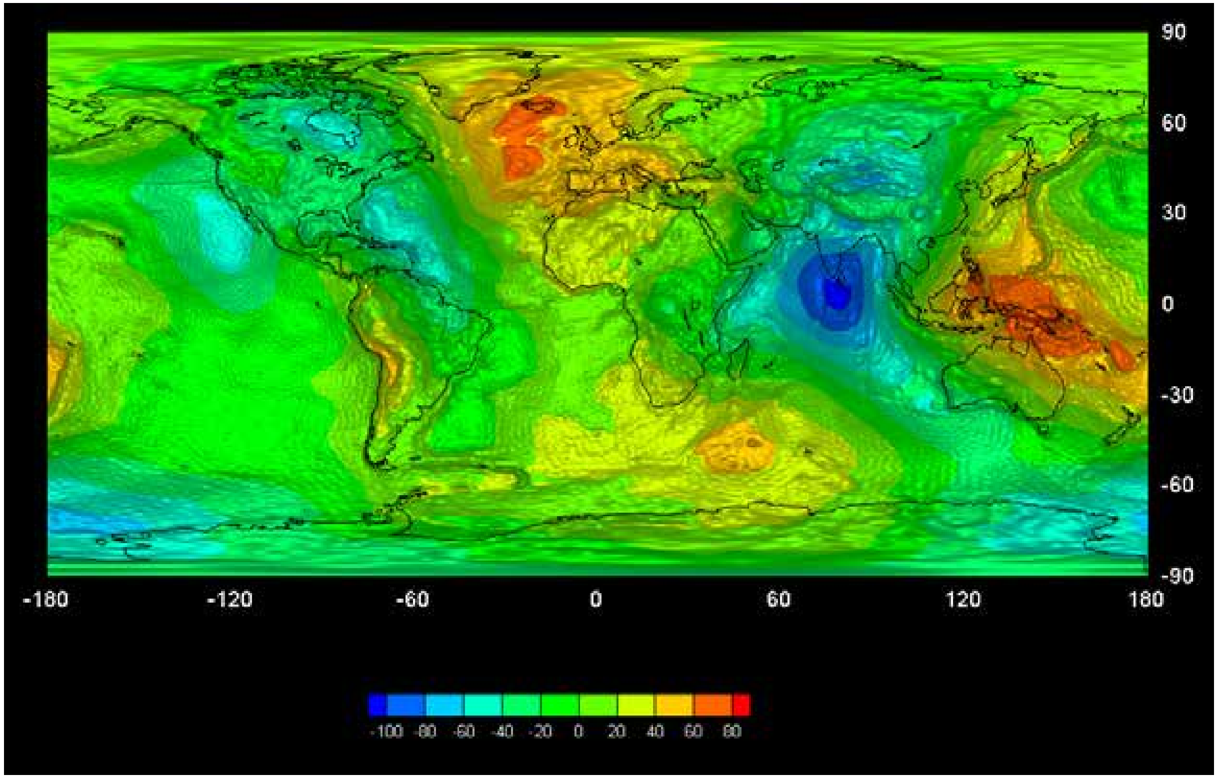
$T_r = \delta V_r$



$T_{rr} = \delta V_{rr}$



introduction



$$f(\theta,\lambda) = k\sum_{l=0}^{\infty}\sum_{m=0}^{l}\bar{P}_{lm}(\cos\theta)\left(\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda\right)$$

$$\begin{Bmatrix}\bar{C}_{lm}\\ \bar{S}_{lm}\end{Bmatrix} = \frac{1}{4\pi}\frac{1}{k}\int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} f(\theta,\lambda)\bar{P}_{lm}(\cos\theta)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\sin\theta\, d\lambda\, d\theta$$

introduction

Observations:

- surface spherical harmonic (=SH)-base functions are a complete set of ”building blocks” for the representation of functions on a sphere
- two coordinates and two indices (degree l and order m)
- (some) analogy the 2D-FT therefore sometimes referred to as FOURIER analysis on a sphere
- for finite maximum degree L: (instead of infinity) still best possible approximation
- meridian lines converge towards the poles (sphere versus torus)

$$f(\theta,\lambda) = k\sum_{l=0}^{\infty}\sum_{m=0}^{l}\bar{P}_{lm}(\cos\theta)\left(\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda\right)$$

$$\begin{Bmatrix}\bar{C}_{lm}\\ \bar{S}_{lm}\end{Bmatrix} = \frac{1}{4\pi}\frac{1}{k}\int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} f(\theta,\lambda)\bar{P}_{lm}(\cos\theta)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\sin\theta\, d\lambda\, d\theta$$

representation of functions on a sphere

As series of spherical harmonic (SH-) functions

surface spherical harmonic functions:

$$Y_{\ell m}(\theta,\lambda)=\bar{P}_{\ell m}(\cos\theta)\,e^{im\lambda}$$

$$with:\ \bar{P}_{\ell m}(\cos\theta)=\begin{cases} N_{\ell m}P_{\ell m}(\cos\theta) & m\geq 0 \\ (-1)^m P_{\ell,-m}(\cos\theta) & m<0 \end{cases}$$

and

$$N_{\ell m}=(-1)^m\sqrt{(2\ell+1)\frac{(\ell-m)!)}{(\ell+m)!}}$$

The closed triplet: synthesis, orthogonality and analysis:

$$f(\theta,\lambda)=k\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}K_{\ell m}Y_{\ell m}(\theta,\lambda)$$

$$\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi}Y_{\ell m}(\theta,\lambda)Y_{\ell' m'}(\theta,\lambda)\sin\theta\,d\lambda\,d\theta=\delta_{\ell\ell'}\delta_{mm'}$$

$$K_{\ell m}=\frac{1}{4\pi}\frac{1}{k}\iint_{\sigma}f(\theta,\lambda)Y_{\ell m}(\theta,\lambda)\,d\sigma$$

representation of periodic functions in the plane

Example: 2D-FOURIER-series

$$f(x,y) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} f_{kl} \exp\left(i\left(kx + ly\right)\right)$$

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{1}{2\pi}\int_{-\pi}^{+\pi} \left[\exp(i(k - k')x)\right] dx \left[\exp\left(i\left(l - l'\right)y\right)\right] dy = \delta_{kk'}\delta_{ll'}$$

$$f_{kl} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x,y) \exp\left(-i\left(kx + ly\right)\right) dy\,dx$$

representation of functions on a sphere

Classical (non-complex) notation

$$f(\theta,\lambda) = k\sum_{l=0}^{\infty}\sum_{m=0}^{l} \bar{P}_{lm}(\cos\theta)\left(\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda\right)$$

$$\begin{Bmatrix} \bar{C}_{lm} \\ \bar{S}_{lm} \end{Bmatrix} = \frac{1}{4\pi}\frac{1}{k}\int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} f(\theta,\lambda)\bar{P}_{lm}(\cos\theta)\begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix}\sin\theta \, d\lambda \, d\theta$$

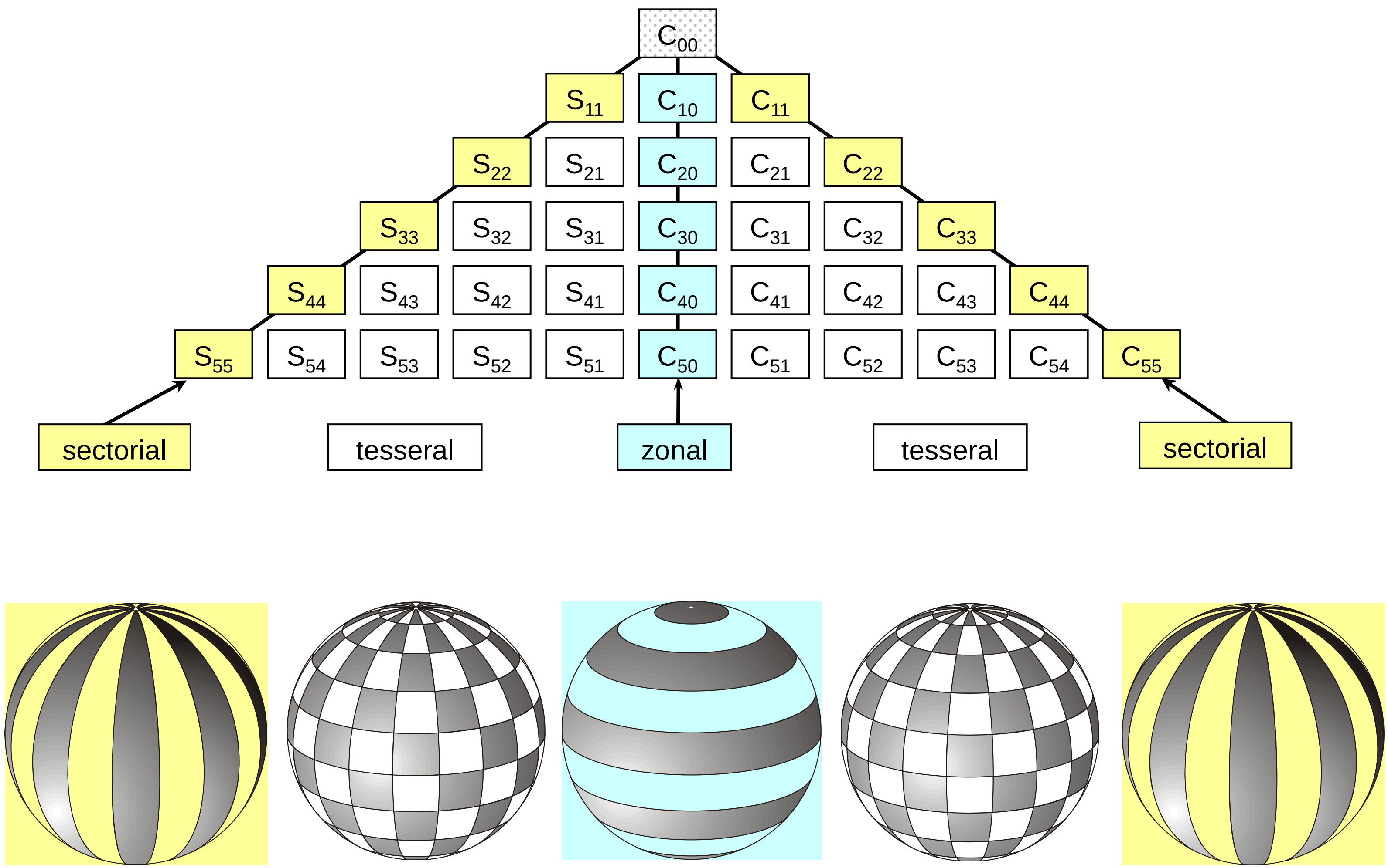
Connection between classical and complex:

$$K_{lm} = \begin{Bmatrix} (-1)^m\left(\bar{C}_{lm} - i\bar{S}_{lm}\right)/\sqrt{2} & m > 0 \\ \bar{C}_{lm} & m = 0 \\ \left(\bar{C}_{lm} + i\bar{S}_{lm}\right)/\sqrt{2} & m < 0 \end{Bmatrix}$$

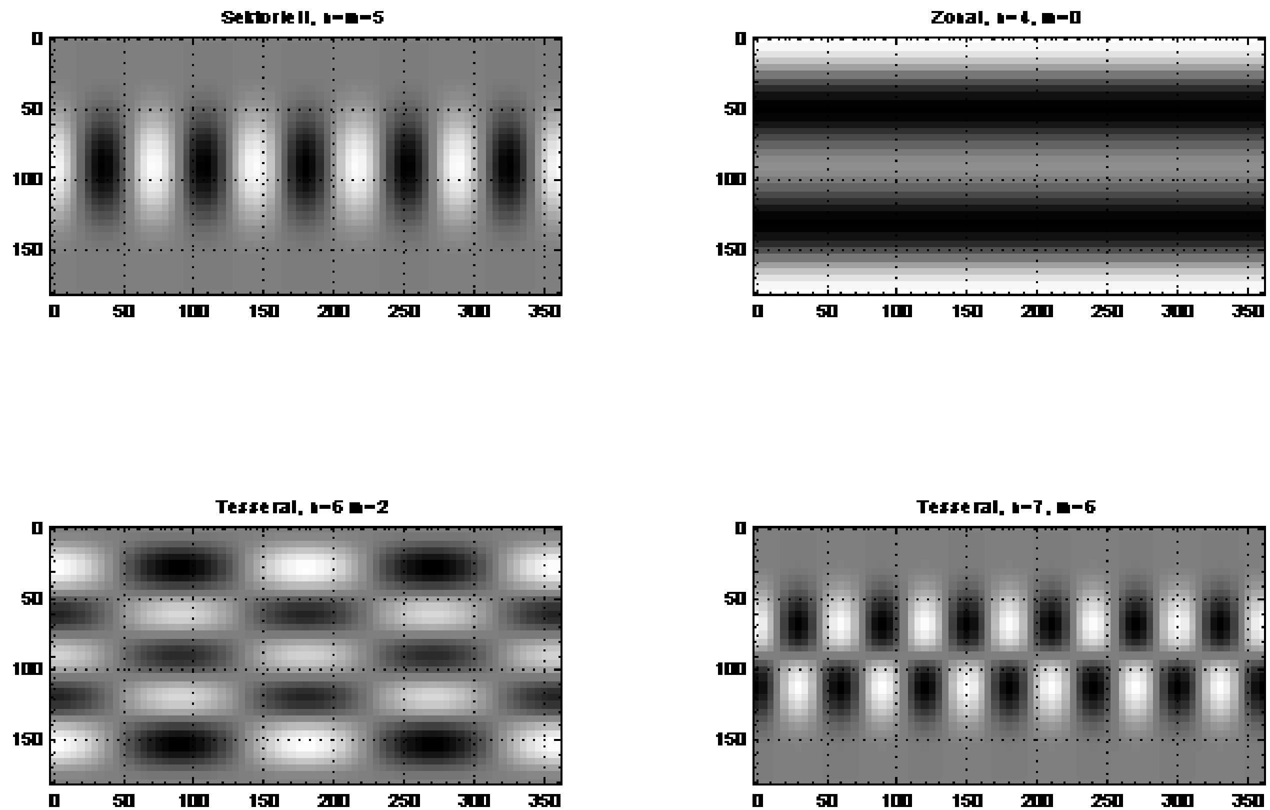
and

$$K_{lm} = (-1)^m K^{*}_{l,-m}$$

series representation of functions on a sphere



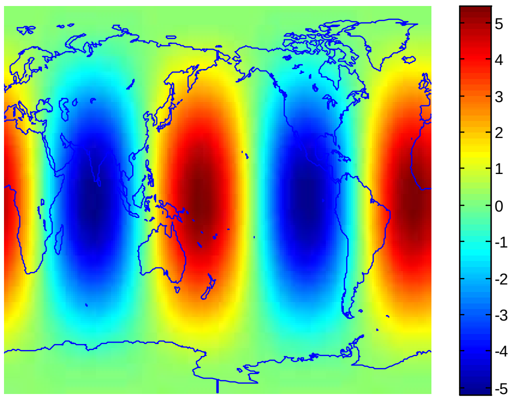
surface spherical harmonic functions: $Y_{lm}(\varphi, \lambda) = \bar{P}_{l|m|}(\varphi) \begin{cases} \cos m\lambda \\ \sin m\lambda \end{cases}$


Sektoriell, n=m=5
Zonal, n=4, m=0
Tesseral, n=6 m=2
Tesseral, n=7, m=6

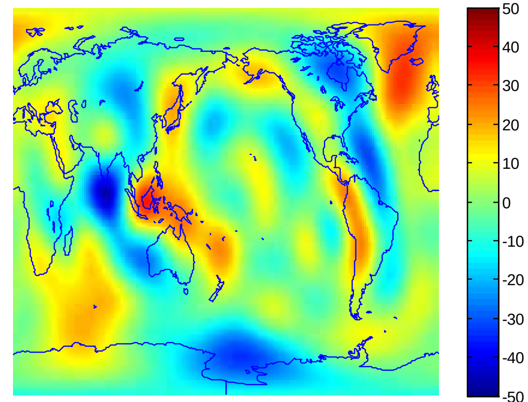
series representation of functions on a sphere

the higher the degree and order of the series, the more details

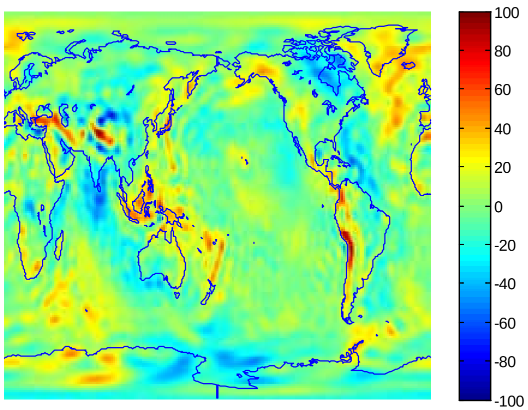
l = 0:2



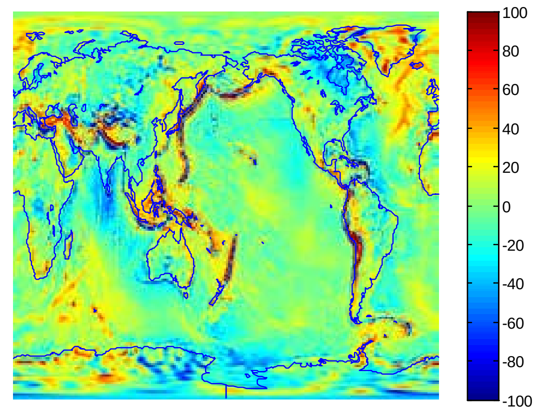
l = 0:10



l = 0:50

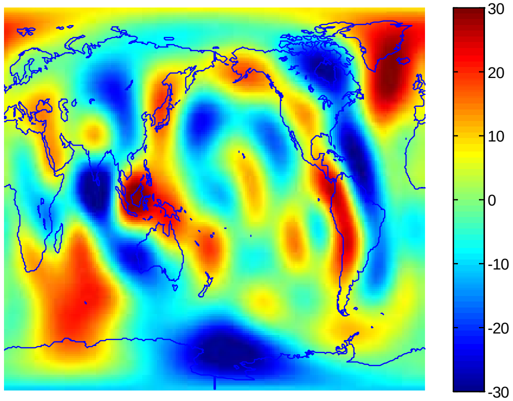


l = 0:150

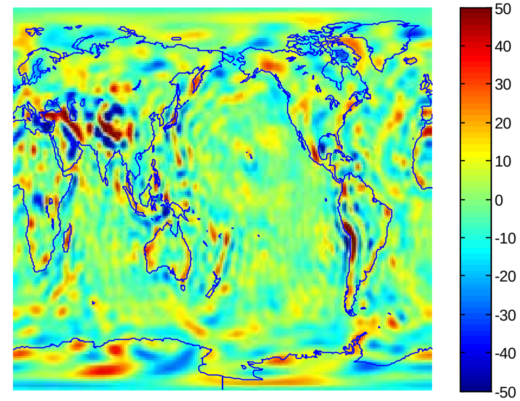


series representation of functions on a sphere

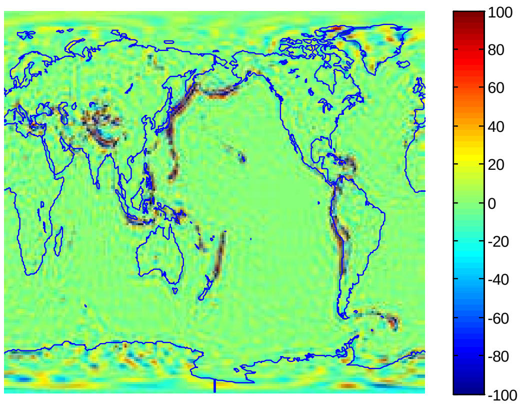
l = 0:10 - 0:2 = 3:10



l = 0:50 - 0:10=11:50



l = 0:150 - 0:50=51:150



Rule of thumb:

$$s = \frac{20000\text{km}}{L_{max}}$$

- Why is it a triangle?
- How many coefficients up to $l = L$?
- Base functions on a sphere: how do they look like?
- Coefficients are weights of base functions
- Why is it a double sum and not a double integral?

FOURIER representation

Given function is:

discrete periodic	discrete non-periodic
continuous periodic	continuous non-periodic

Fourier transformation is:

discrete FOURIER-series

periodic discrete	periodic continuous
non-periodic discrete	non-periodic continuous

classical FOURIER-series
(= Sinus-Cosinus-series)

FOURIER-transform (FT)

topology of a sphere: twofold periodic

LEGENDRE polynomials

a short introduction:

from LEGENDRE-polynomials

to associated LEGENDRE functions

to spherical harmonic functions

functions f(t) between t= -1 and t=+1 can be expanded

into a (complete) LEGENDRE-series:

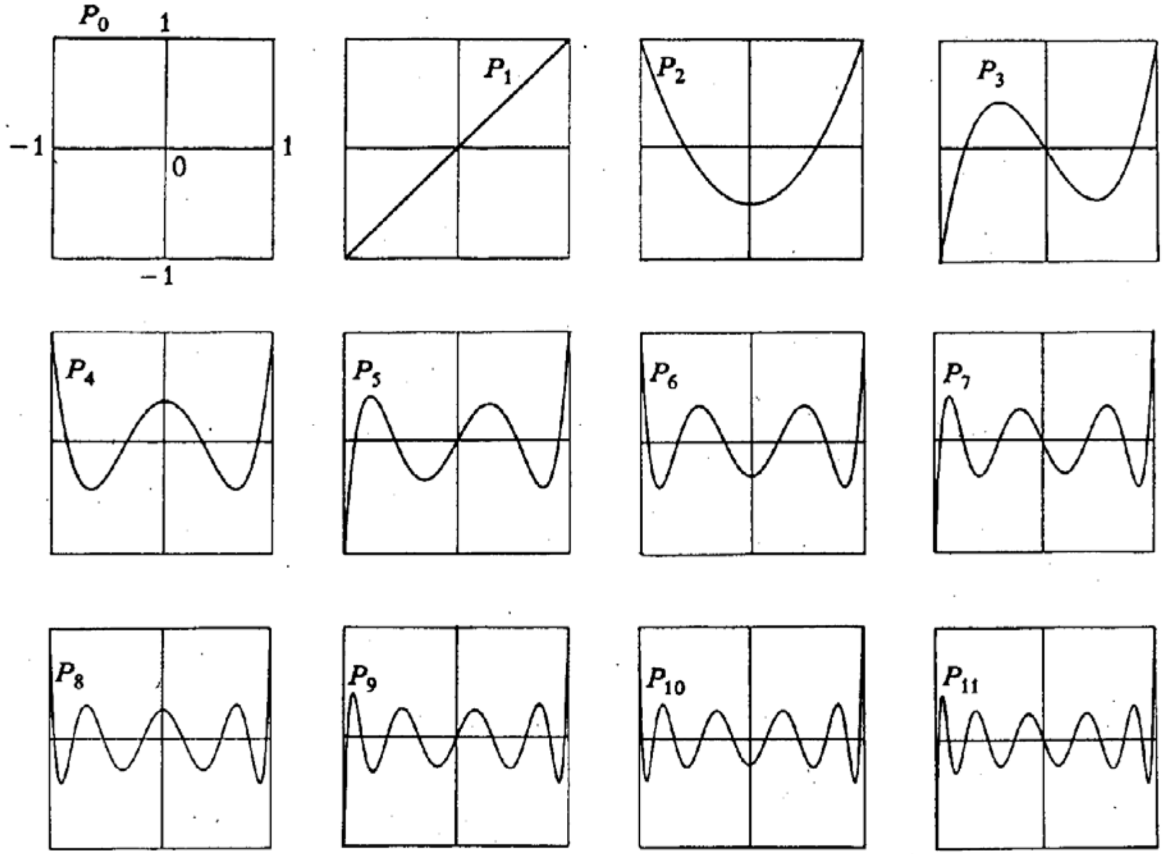
The closed triplet: synthesis, orthogonality and analysis:

$$f(t) = \sum_{l=0}^{\infty} f_l P_l(t)$$ synthesis

$$\int_{t=-1}^{+1} P_l(t) P_{l'}(t)\, dt = \frac{2}{2l+1} \delta_{ll'}$$ orthogonality

$$\int_{t=-1}^{+1} f(t) P_l(t)\, dt = \frac{2}{2l+1} f_l$$ analysis

LEGENDRE polynomials



from: Jänich,1990

LEGENDRE polynomials

Formula by Rodriguez:

$$P_l(t) = \frac{1}{2^l\, l!} \frac{d^l}{dt^l} \left(t^2 - 1\right)^l$$

Characteristic differential equation:

$$P_l(1) = 1$$

$$(1 - t^2) P_l''(t) - 2t P_l'(t) + l(l+1) P_l(t) = 0$$

Orthogonality:

$$\int_{t=-1}^{+1} P_l(t) P_{l'}(t)\, dt = \frac{2}{2l+1} \delta_{ll'}$$

Recursive computation:

$$(l+1) P_{l+1}(t) = (2l+1) t P_l(t) - l P_{l-1}(t)$$

$$P_l'(t)(1 - t^2) = l\left(t P_l(t) - P_{l-1}(t)\right)$$

LEGENDRE polynomials in theta

$$\textit{with} \quad t = \cos\theta :$$

$$\{t \mid -1 \le t \le +1\} \Leftrightarrow \{\theta \mid 0 \le \theta \le \pi\}$$

$$\textit{or} \quad \{\theta \mid \textit{north pole} \le \theta \le \textit{south pole}\}$$

And the closed triplet: synthesis, orthogonality and analysis turns into:

$$f(\theta) = \sum_{l=0}^{\infty} f_l P_l(\theta)$$

$$\int_{\theta=0}^{\pi} P_l(\theta) P_{l'}(\theta) \sin\theta \, d\theta = \frac{2}{2l+1} \delta_{ll'}$$

$$\int_{\theta=0}^{\pi} f(\theta) P_l(\theta) \sin\theta \, d\theta = \frac{2}{2l+1} f_l$$

The „north pole“ may be chosen anywhere on the sphere

LEGENDRE polynomials in theta

One very fundamental LEGENDRE-series:

$$\frac{1}{d_{PQ}} = \frac{1}{R}\sum_{l=0}^{\infty}\left(\frac{R}{r_P}\right)^{l+1} P_l\left(\cos\theta_{PQ}\right)$$

and

$$d_{PQ} = \sqrt{R^2 + r_P^{\,2} - 2Rr\cos\theta_{PQ}}$$

(think of NEWTON‘s inverse squared distance law)

P

d

r Q

θ R

O

Example: P at satellite, Q on the earth’s surface,
O the earth’s centre, R = earth radius

associated LEGENDRE-functions

We had:

$$P_l(t) = \frac{1}{2^l\, l!} \frac{d^l}{dt^l} \left(t^2 - 1\right)^l$$

Formula by Rodriguez:

$$P_{lm}(t) = \left(1 - t^2\right)^{\frac{m}{2}} P_l^{(m)}(t) = \left(1 - t^2\right)^{\frac{m}{2}} \frac{1}{2^l\, l!} \frac{d^{l+m}}{dt^{l+m}} \left(t^2 - 1\right)^l$$

where $P_l^{(m)}$ *is the* m*-th derivative*

Characteristic differential equation:

$$(1 - t^2) P''_{lm}(t) - 2t P'_{lm}(t) + \left[l(l+1) - \frac{m^2}{1 - t^2} \right] P_{lm}(t) = 0$$

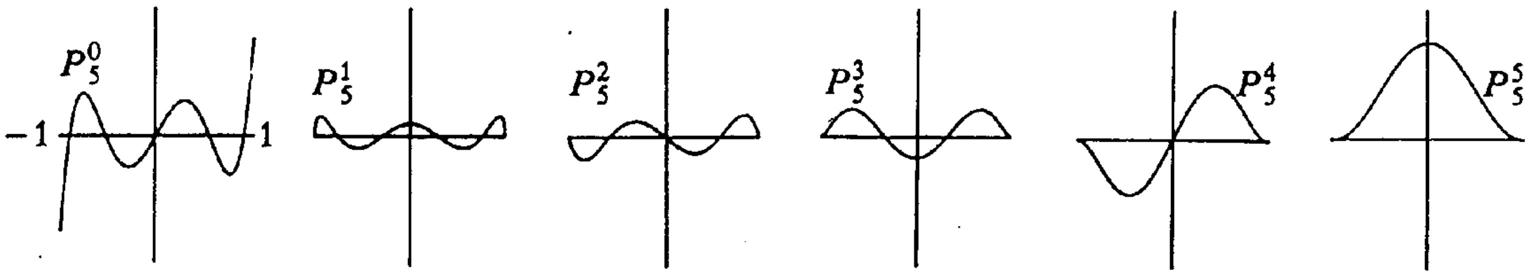
Some properties:

$$P_{l0}(t) = P_l(t)$$

$$P_{lm}(t) = 0 \quad \text{for} \quad m > l$$

$$P_{ll}(t) = \left(1 - t^2\right)^{\frac{m}{2}} \times \text{const.}$$

associated LEGENDRE-functions



From: Jänich, 1990

Some properties:

for $l - m$ even $\Rightarrow$ P_{lm} even

for $l - m$ odd $\Rightarrow$ P_{lm} odd

P_{lm} has $l - m$ zeros in $\{t \mid -1 < t < +t\}$

for $m > 0 : P_{lm} = 0$ at $t = -1$ and $t = +1$

associated LEGENDRE-functions

Orthogonality:

$$\int_{-1}^{+1} P_{lm}(t)P_{l'm}(t)dt = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}$$

$$\left[\int_{-1}^{+1}\left(1-t^2\right)^{-1} P_{lm}(t)P_{lm'}(t)dt = 0 \quad \text{if } m \neq m'\right]$$

Recursion formulas:

$$P_{ll}(t) = (2l-1)\sqrt{1-t^2}P_{l-1,l-1}(t)$$

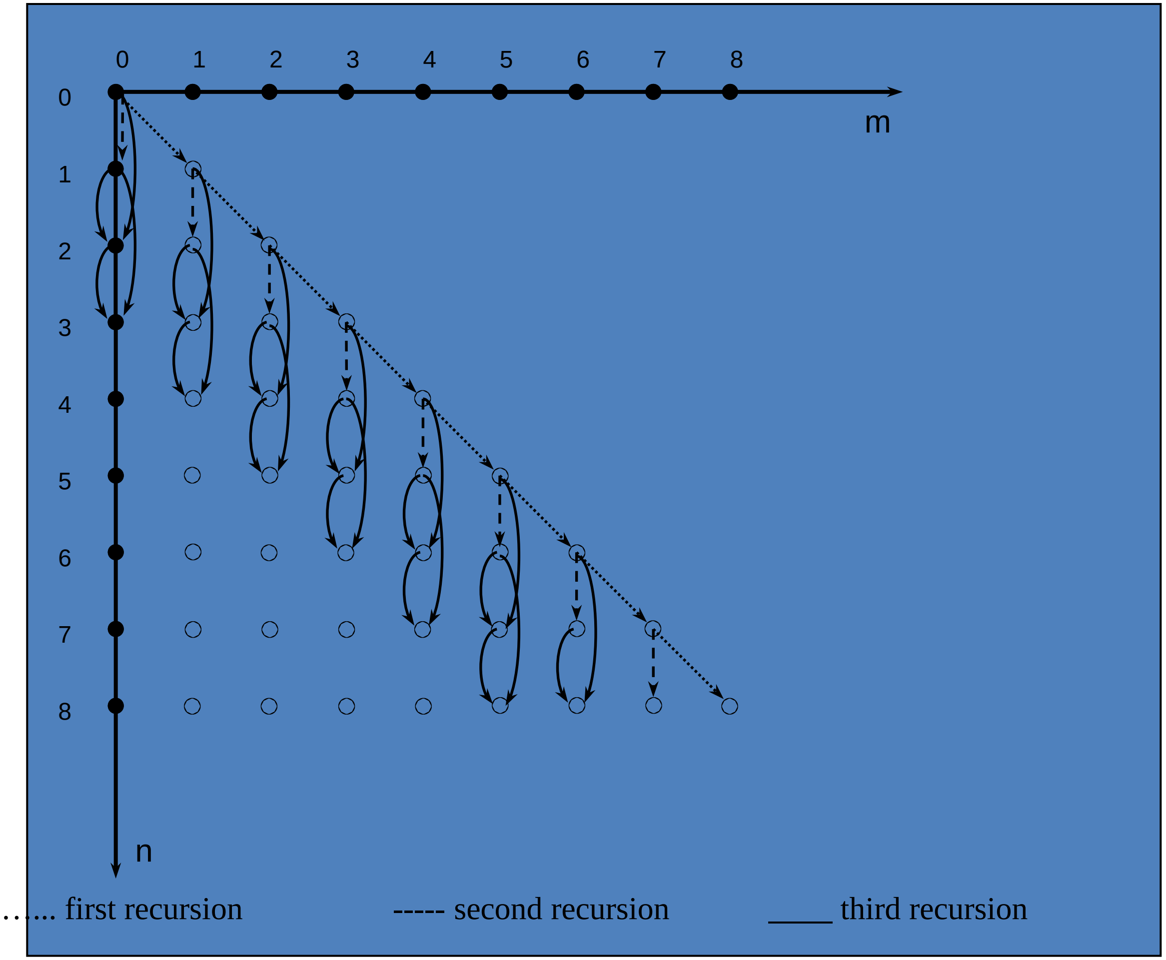
$$P_{l,l-1}(t) = (2l-1)tP_{l-1,l-1}(t)$$

$$P_{lm}(t) = \frac{2l-1}{l-m}tP_{l-1,m}(t) - \frac{l+m-1}{l-m}P_{l-2,m}(t)$$

also:

$$\left(1-t^2\right)P'_{lm}(t) = -ltP_{lm}(t) + (l+m)P_{l-1,m}(t)$$

associated LEGENDRE-functions



back to representation in surface spherical harmonics

Classical (non-complex) notation

$$f(\theta,\lambda) = k\sum_{l=0}^{\infty}\sum_{m=0}^{l}\bar{P}_{lm}(\cos\theta)\left(\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda\right)$$

$$\begin{Bmatrix}\bar{C}_{lm}\\ \bar{S}_{lm}\end{Bmatrix} = \frac{1}{4\pi}\frac{1}{k}\int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} f(\theta,\lambda)\bar{P}_{lm}(\cos\theta)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\sin\theta\, d\lambda\, d\theta$$

Can be written as (almost) like a FOURIER-series:

$$f(\theta,\lambda) = k\sum_{m=0}^{\infty}\left(C_m(\theta)\cos m\lambda + S_m(\theta)\sin m\lambda\right)$$

with $C_m(\theta) = \sum_{l=m}^{\infty}\bar{C}_{lm}\bar{P}_{lm}(\theta)$ and $S_m(\theta) = \sum_{l=m}^{\infty}\bar{S}_{lm}\bar{P}_{lm}(\theta)$

step1: $\begin{Bmatrix}C_m(\theta)\\ S_m(\theta)\end{Bmatrix} = \frac{1}{\pi}\int_0^{2\pi} f(\theta,\lambda)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix} d\lambda$ for $m \neq 0$

step 2: $\begin{Bmatrix}\bar{C}_{lm}\\ \bar{S}_{lm}\end{Bmatrix} = \frac{1}{4}\int_0^{\pi}\begin{Bmatrix}C_m(\theta)\\ S_m(\theta)\end{Bmatrix}\bar{P}_{lm}(\theta)\sin\theta\, d\theta$

and

$C_{m=0}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta,\lambda)\, d\lambda$ and $\bar{C}_{00} = \frac{1}{2}\int_0^{\pi} C_{m=0}(\theta)\sin\theta\, d\theta$

filtering in the time and spectral domain

Convolution with stationary filter functions:

$$f(t) * w(t-t') = g(t)$$

⇔

FOURIER Transformation

⇔

$$F(u) \cdot W(u) = G(t)$$

Convolution in the time domain corresponds
to multiplication in the spectral domain,
and vice versa

filtering on a sphere and in the SH-spectral domain

Convolution with an stationary and isotropic filter functions:

$$f(\theta,\lambda) * w(\psi) = g(\theta',\lambda')$$

$\Leftrightarrow$ Spherical Harmonic Transformation $\Leftrightarrow$

$$K_{lm} \cdot W_l = G_{lm}$$

Convolution in the time domain corresponds to multiplication in the spectral domain, and vice versa

isotropic filter functions

Pellinen Function: spherical equivalent of a box function

$$B(\psi) = \begin{cases} \dfrac{1}{2\pi(1-\cos\Psi)} & \text{and } \psi \le \Psi \\ 0 & \text{and } \psi > \Psi \end{cases}$$

or as Legendre series

$$B(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} b_n P_n(\cos\psi)$$

Expansion into Legendre polynomials:

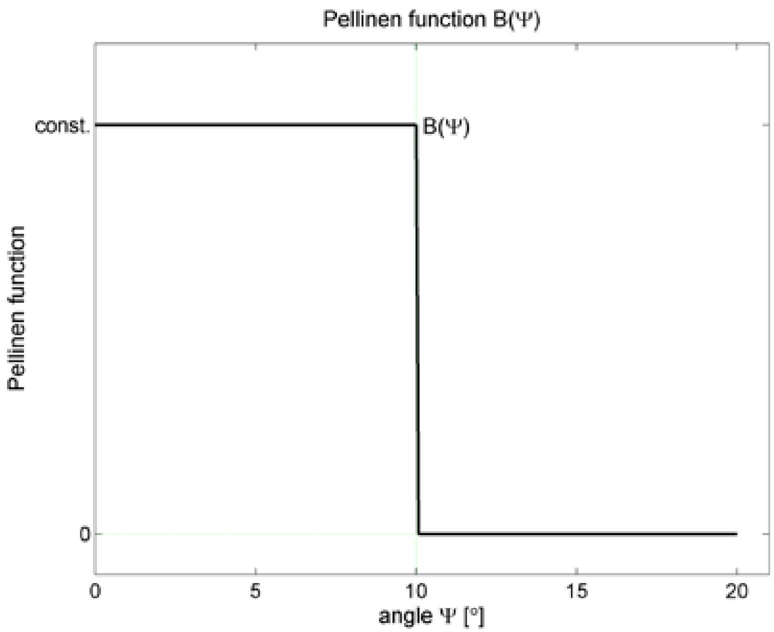
$$b_n = 2\pi \int_{\psi=0}^{\pi} B(\psi) P_n(\cos\psi)\sin\psi\, d\psi = \frac{1}{1-\cos\Psi} \int_{\psi=0}^{\Psi} P_n(\cos\psi)\sin\psi\, d\psi$$

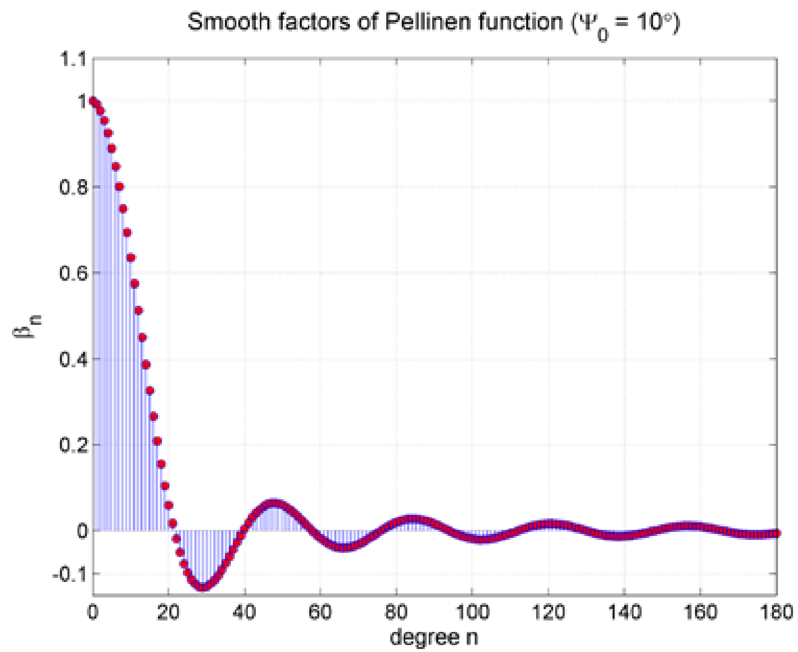
$$= \frac{1}{1-\cos\Psi} \frac{1}{2n+1} \left[P_{n-1}(\cos\Psi) - P_{n+1}(\cos\Psi) \right]$$

Or approximately (Sjöberg L,B.G.,1980):

$$b_n = \frac{2n-1}{n+1}\cos\Psi \cdot b_{n-1} - \frac{n-2}{n+1} b_{n-2} \quad \text{with } b_0 = 1 \text{ and } b_1 = \tfrac{1}{2}(1+\cos\Psi)$$

isotropic filter functions





Pellinen-function

isotropic filter functions

Jekeli Function: spherical equivalent of a Gauss function

(Jekeli C, OSU327, 1981):

$$w(\psi) = \frac{\beta}{2\pi} \frac{\exp\left[-\beta \cdot (1 - \cos\psi)\right]}{1 - \exp(-2\beta)} \qquad where\ \beta = \frac{\ln(2)}{\left(1 - \cos(s/R)\right)}$$

(It is s= full width (arc length on earth sphere) of half value and R = earth radius or Ψ=s/R)

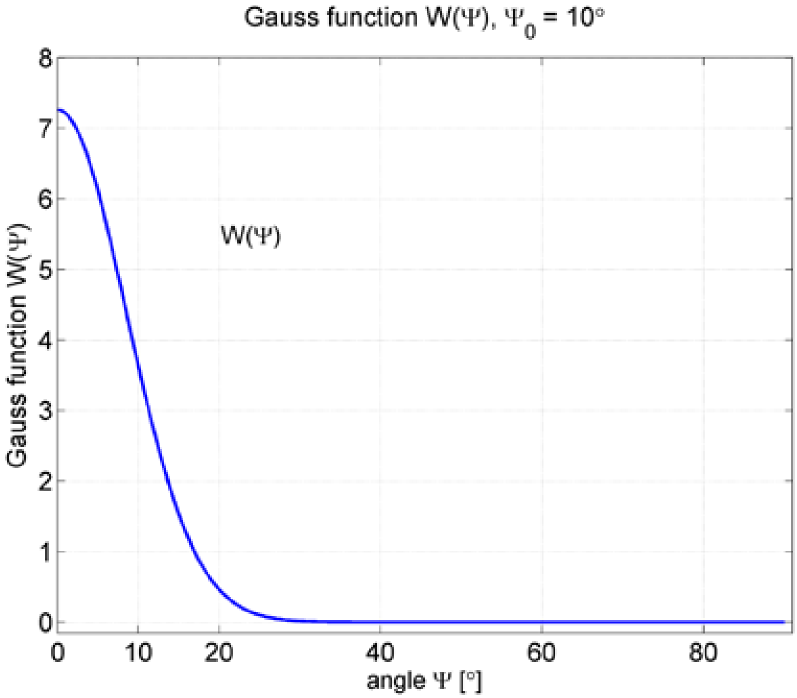
Expansion of w(ψ) into Legendre polynomials:

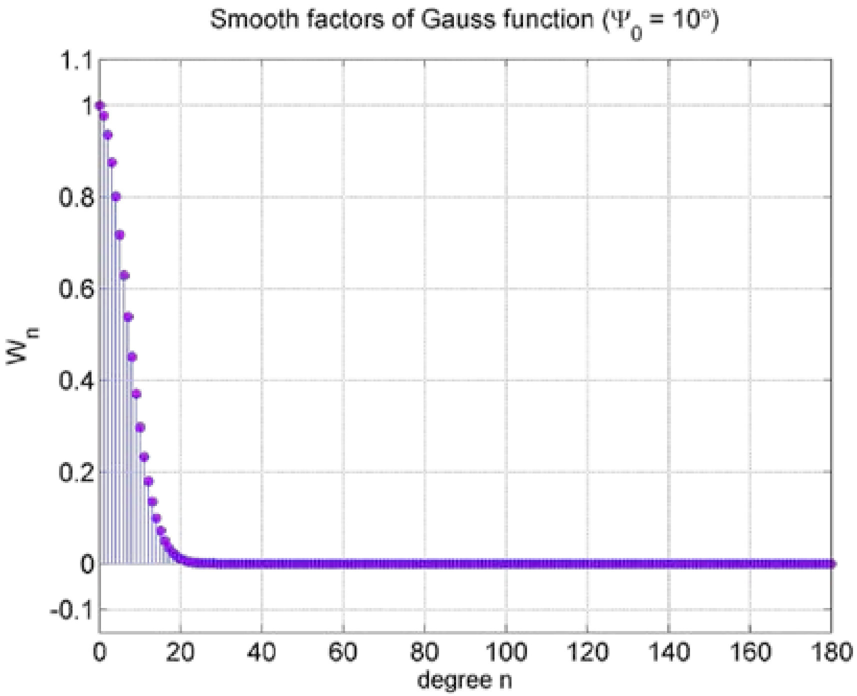
$$W_n = \int_{\psi=0}^{\pi} w(\psi) P_n(\cos\psi) \sin\psi \, d\psi$$

Recursion formulae:

$$W_{n+1} = -\frac{2n+1}{\beta} W_n + W_{n-1} \quad where\ W_0 = \frac{1}{2\pi}\ and\ W_1 = \frac{1}{2\pi}\left[\frac{1+\exp(-2\beta)}{1-\exp(-2\beta)} - \frac{1}{\beta}\right]$$

isotropic filter functions





Jekeli-function

filtering in the time and spectral domain

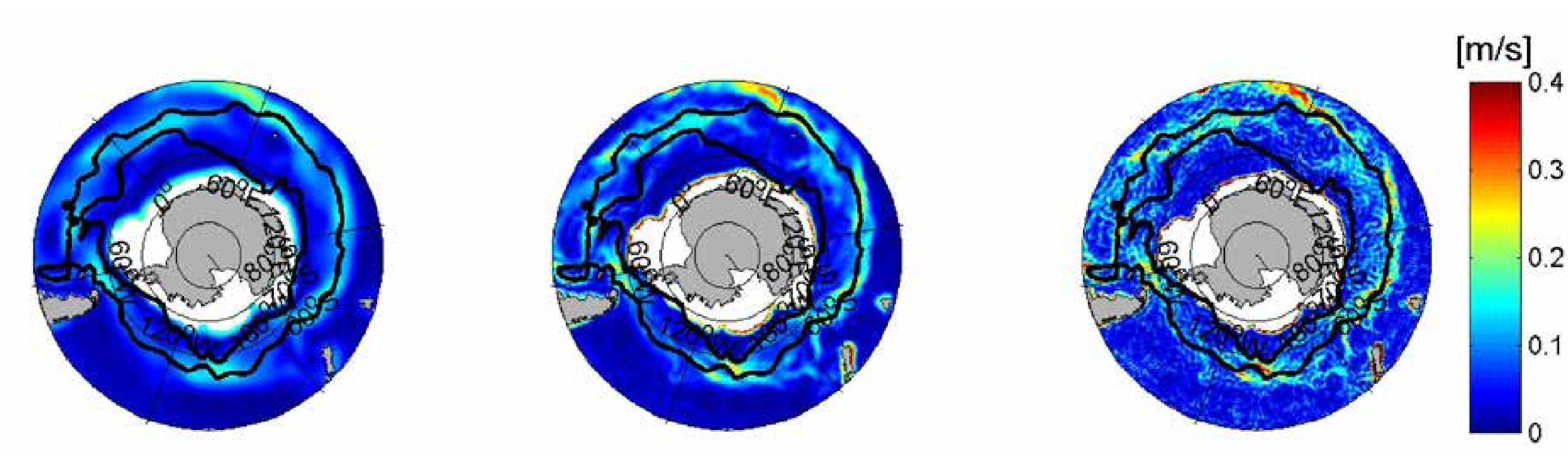
filtered function:

$$f^{W}(\theta,\lambda) = k\sum_{l=0}^{\infty} W_l \sum_{m=0}^{l} \bar{P}_{lm}(\cos\theta)\left(\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda\right)$$

$$\begin{Bmatrix}\bar{C}^{W}_{lm}\\ \bar{S}^{W}_{lm}\end{Bmatrix} = \begin{Bmatrix}W_l\bar{C}_{lm}\\ W_l\bar{S}_{lm}\end{Bmatrix} = \frac{1}{4\pi}\frac{1}{k}\int_{\theta=0}^{\pi}\int_{\lambda=0}^{2\pi} f(\theta,\lambda)\bar{P}_{lm}(\cos\theta)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\sin\theta\, d\lambda\, d\theta$$

Gauss-filtering in the SH-spectral domain

Example:



Geostrophic velocities as derived from the MDT of figure 1and also for D/O=60, D/O=120 and D/O180 (from left to right). Also shown are the fronts derived from oceanographic in-situ data.
Units are meters per second. Source: Albetella, A, 2011

from the surface of a sphere to outer space

Connection from surface to outer/inner space possible,

iff function has certain properties.

The gravitational field V is harmonic outside the earth’s surface.

It fulfills LAPLACE equation.

LAPLACE equation in 3D-Cartesian coordinates:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

LAPLACE equation in 3D-spherical coordinates:

$$r^2 \frac{\partial^2 V}{\partial r^2} + 2r \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \theta^2} + \cot\theta \frac{\partial V}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 V}{\partial \lambda^2} = 0$$

from the surface of a sphere to outer space

Now signal analysis can be extended from surface to outer space

(and satellite observations can be connected with the surface function)

$$V(\theta,\lambda,r)=\frac{k}{R}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{l+1}\sum_{m=-l}^{+l}K_{lm}Y_{lm}(\theta,\lambda)$$

or

$$V(\theta,\lambda,r)=\frac{k}{R}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^{l+1}\sum_{m=0}^{l}\bar{P}_{lm}(\theta)(\bar{C}_{lm}\cos m\lambda+\bar{S}_{lm}\sin m\lambda)$$

solid spherical harmonic functions :

$$r^{-(l+1)}Y_{lm}(\theta,\lambda)=r^{-(l+1)}\bar{P}_{lm}(\cos\theta)e^{im\lambda}$$

$$\textit{with}:\ \bar{P}_{lm}(\cos\theta)=\begin{cases}N_{lm}P_{lm}(\cos\theta) & m\geq 0\\ (-1)^{m}P_{l,-m}(\cos\theta) & m<0\end{cases}$$

$$\textit{and}\quad N_{lm}=(-1)^{m}\sqrt{(2l+1)\frac{(l-m)!}{(l+m)!}}$$

from the surface to outer space

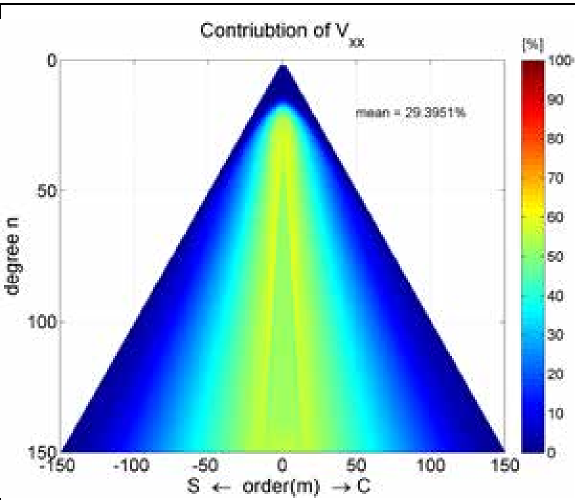
Example: extension of 2D-FOURIER-series to outer space

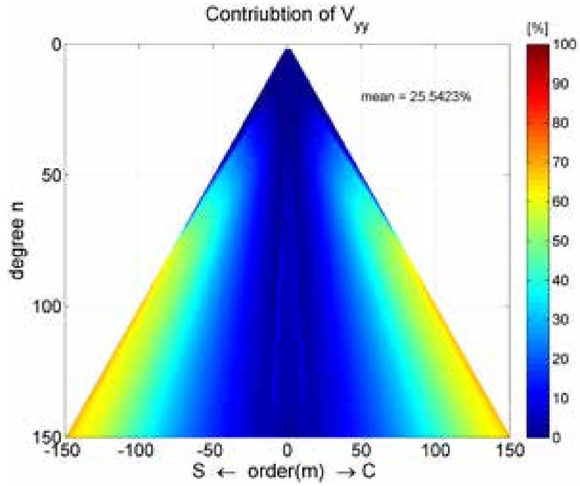
$$f(x,y,z) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} f_{kl} \exp(-mz) \exp(i(kx + ly))$$

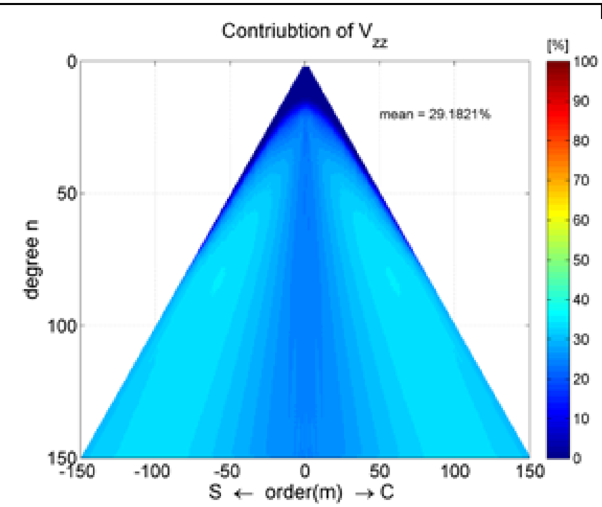
with

$$m^2 = k^2 + l^2$$

example: GOCE analysis of contributions



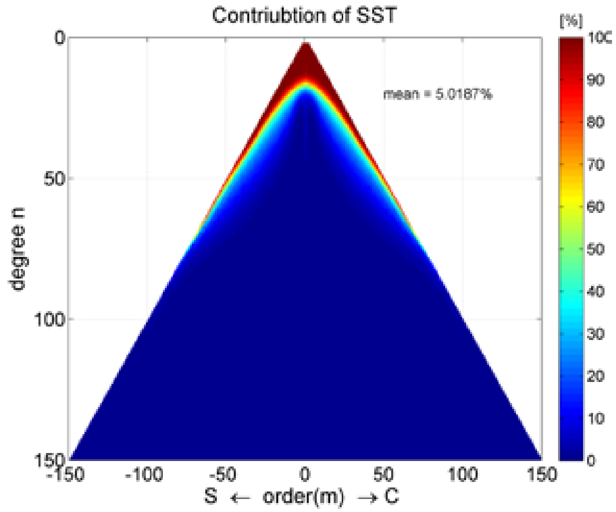


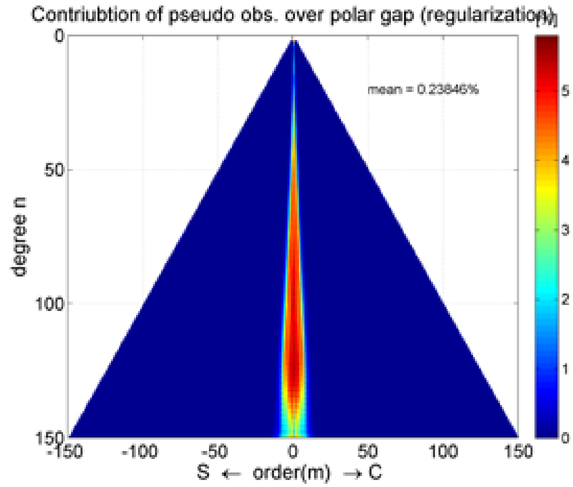


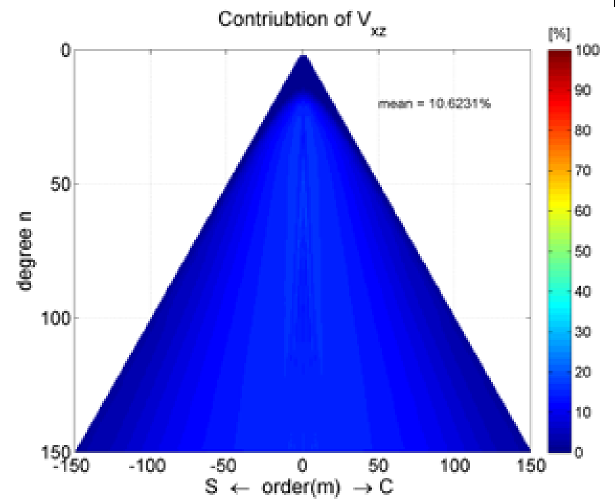
V_{xx} 29.4%

V_{yy} 25.5%

V_{zz} 29.2%







SST(GPS) 5.0%

regularization 0.2%

V_{xz} 10.6%

Weiyong Yi, 2011

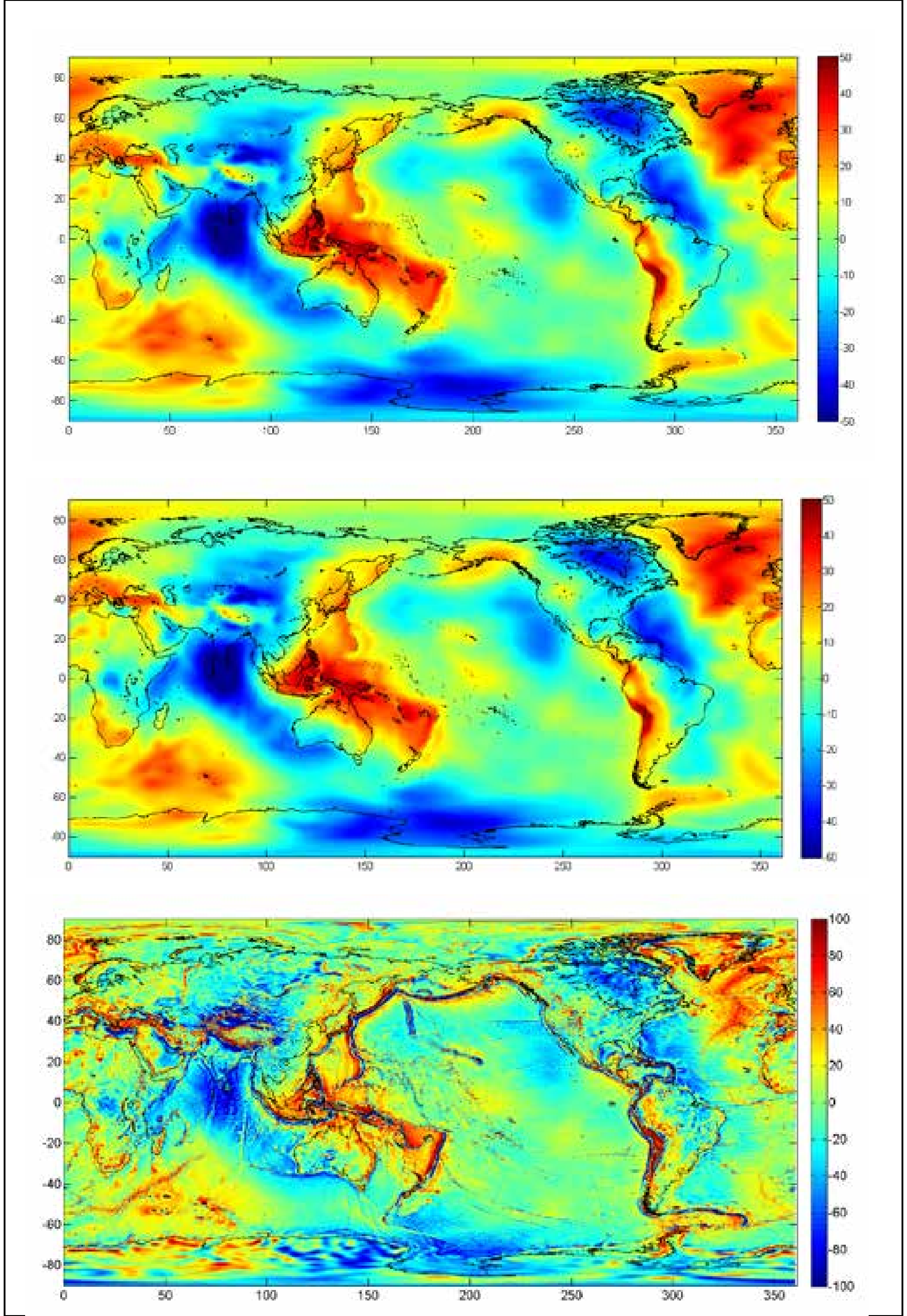
$T_r = \delta V_r$

example: gravity & attenuation with altitude

h = 400km

h = 250km

h = 0km



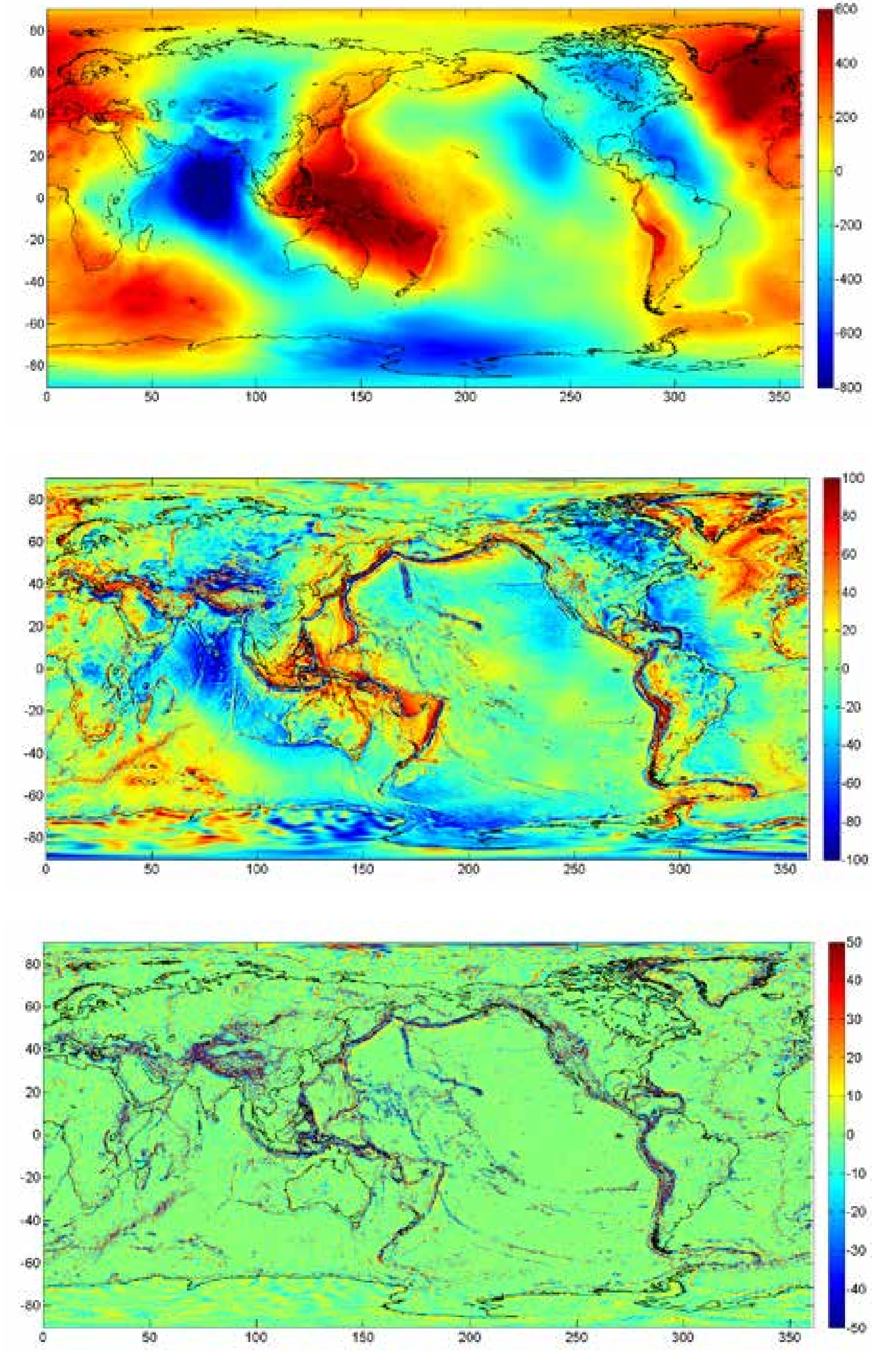
h = 0km

example: potential, first derivative, second derivative

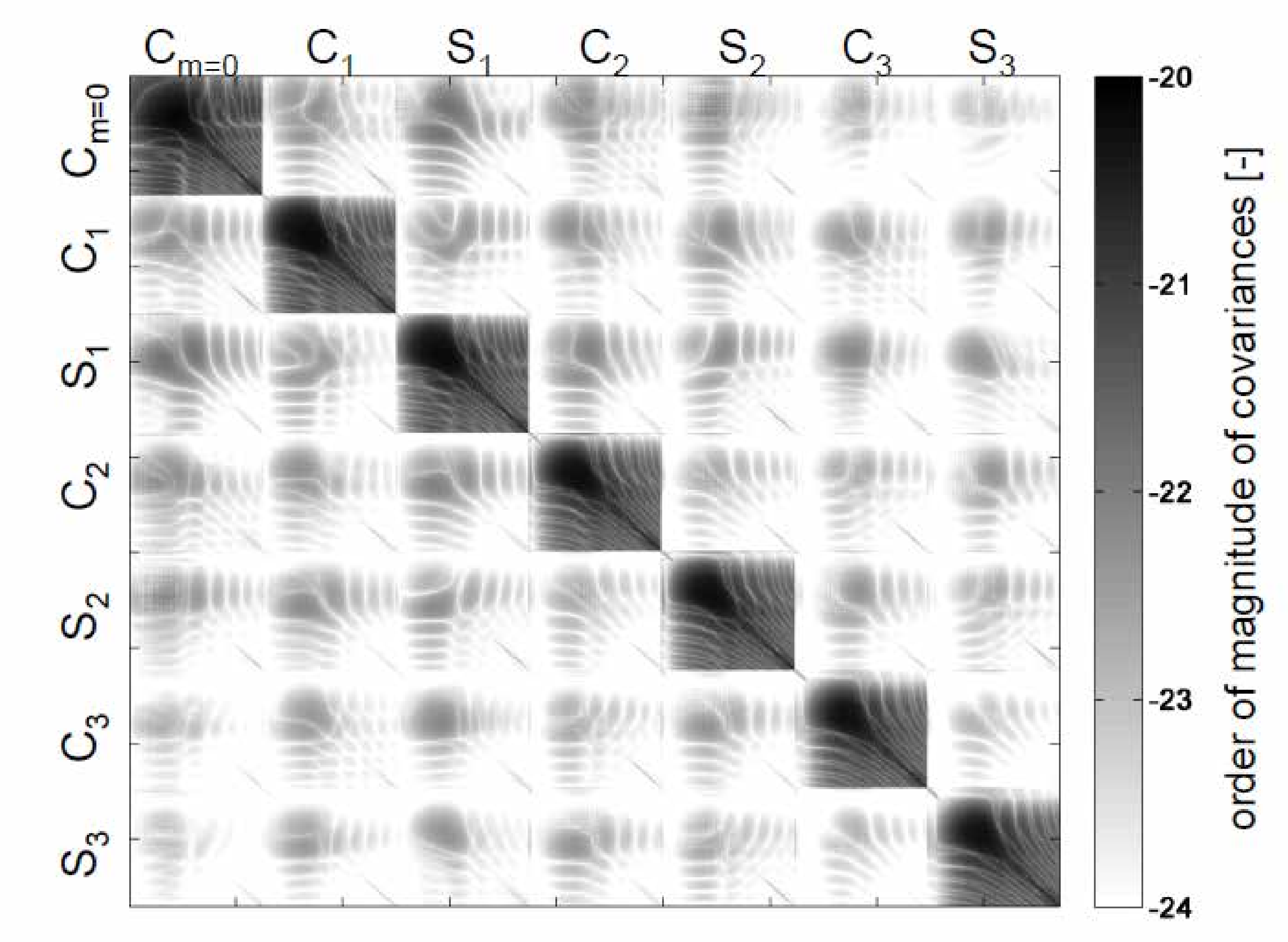
$T = \delta V$

$T_r = \delta V_r$

$T_{rr} = \delta V_{rr}$



example: error variance-covariance propagation



conclusions

-global data analysis on a sphere using surface spherical harmonics
- complete and orthogonal set of spherical base functions
- some similarities with 2D-FOURIER-series
- spherical harmonics are composed of associated LEGENDRE-functions together with trigonometric functions
- can be computed recursively very efficiently
- filtering with stationary and isotropic spherical functions
- spherical harmonics allow separation of PDE’s in spherical coordinates
- extension from earth surface to satellite altitude
- also: connection of function on sphere to time series along orbit
- some textbooks:

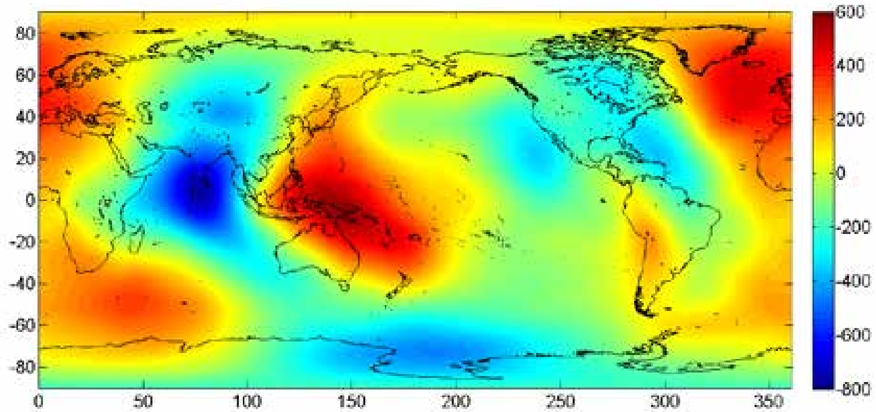
Kellogg OD: Foundations of Potential Theory, Dover, 1953

MacMillan WD: The Theory of the Potential, Dover, 1958

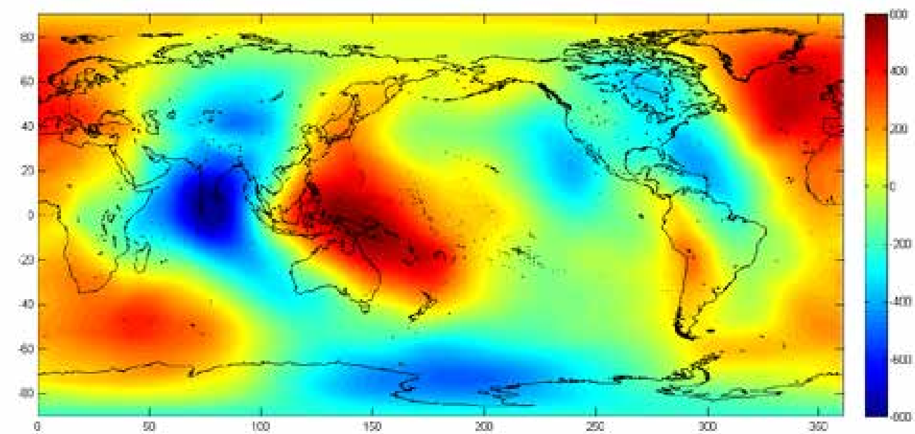
T= δV

example: potential & attenuation with altitude

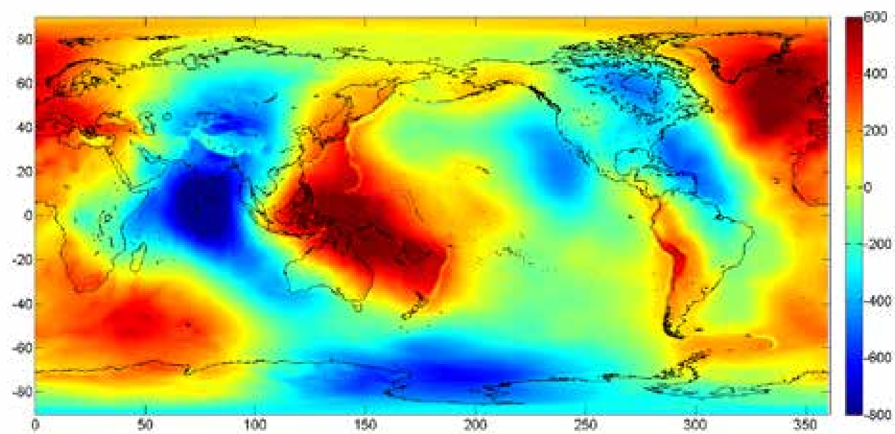
h= 400km



h= 250km



h= 0km

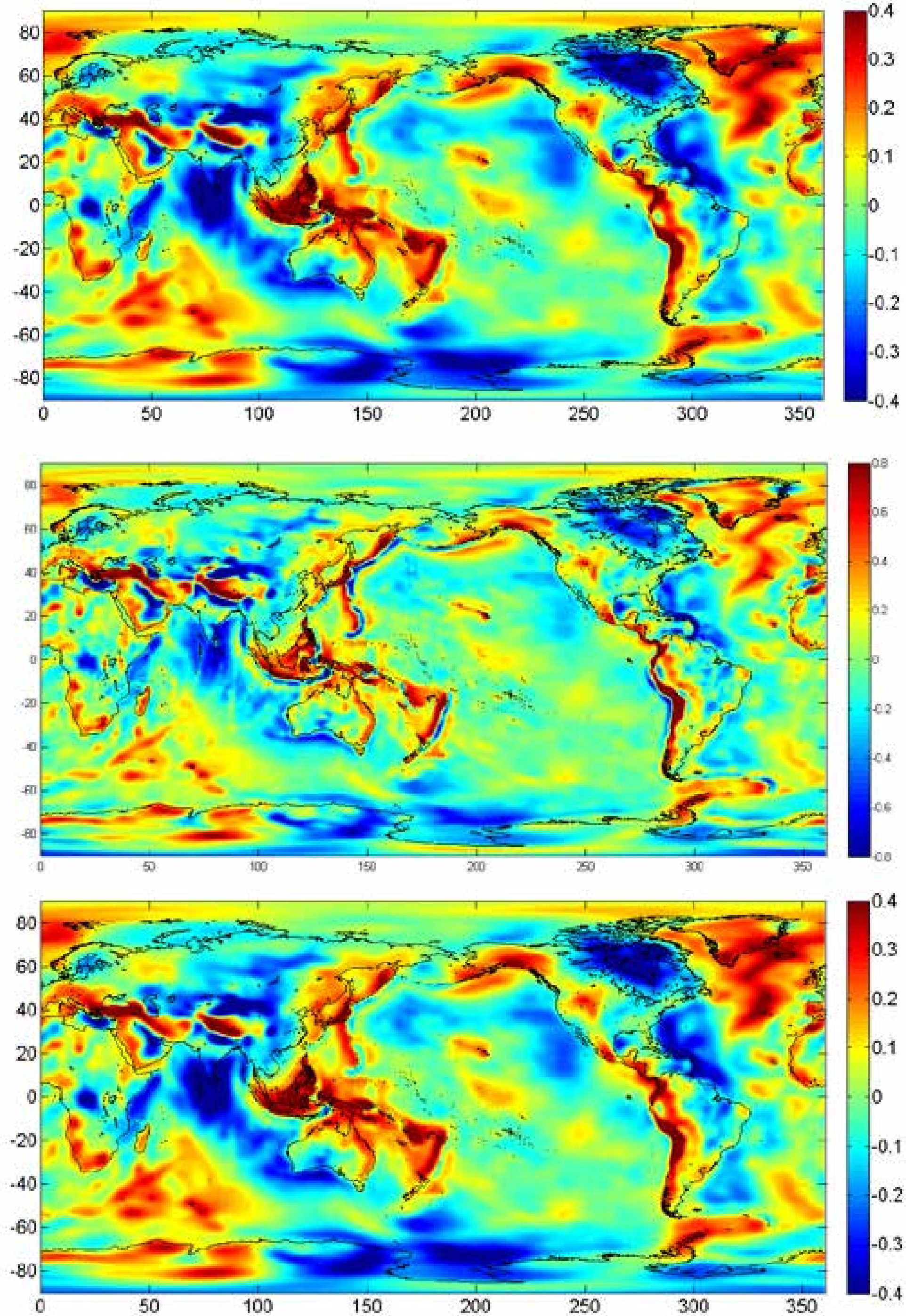


$T_{rr} = \delta V_{rr}$

h= 400km

h= 250km

h= 0km

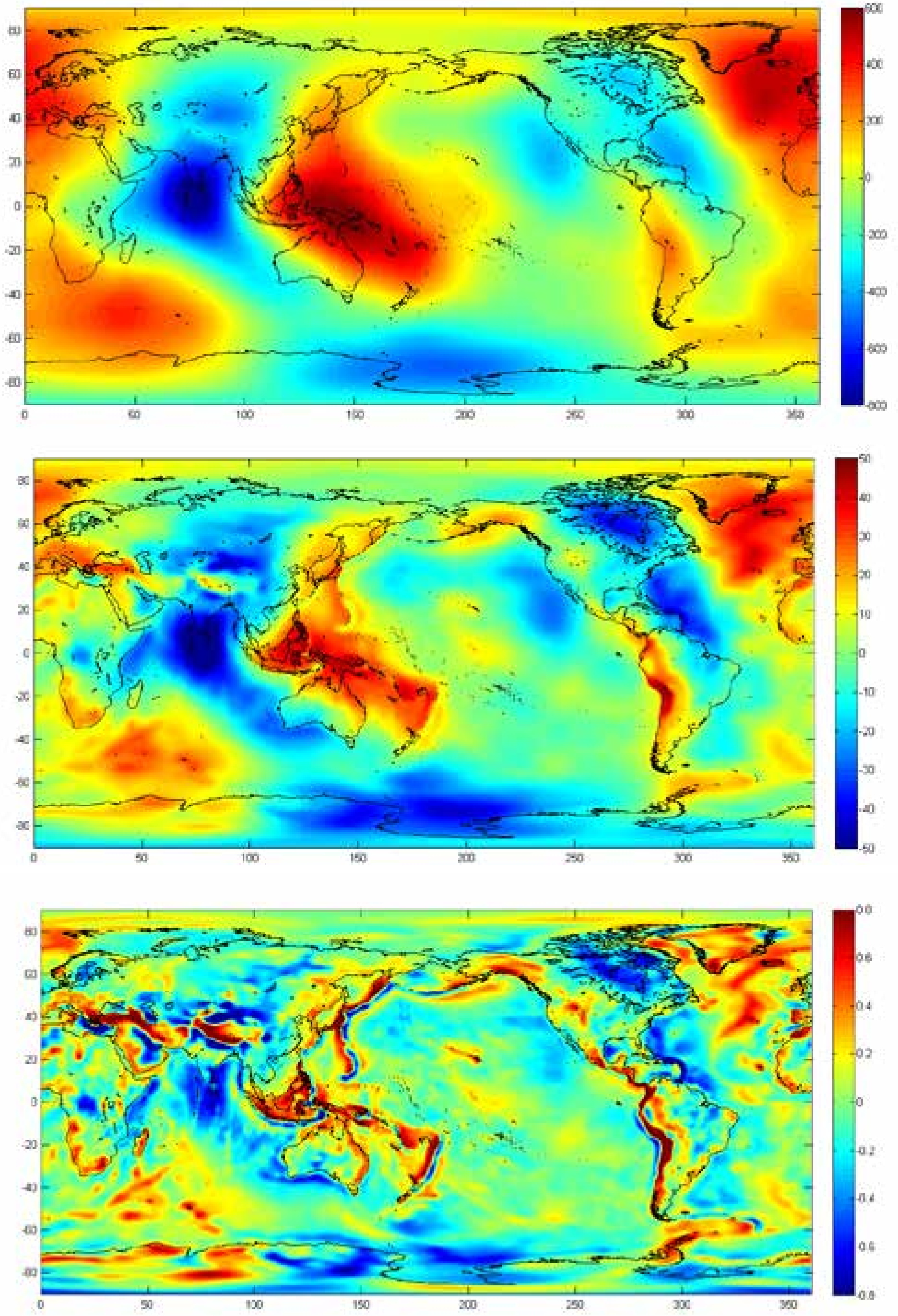


h = 250km

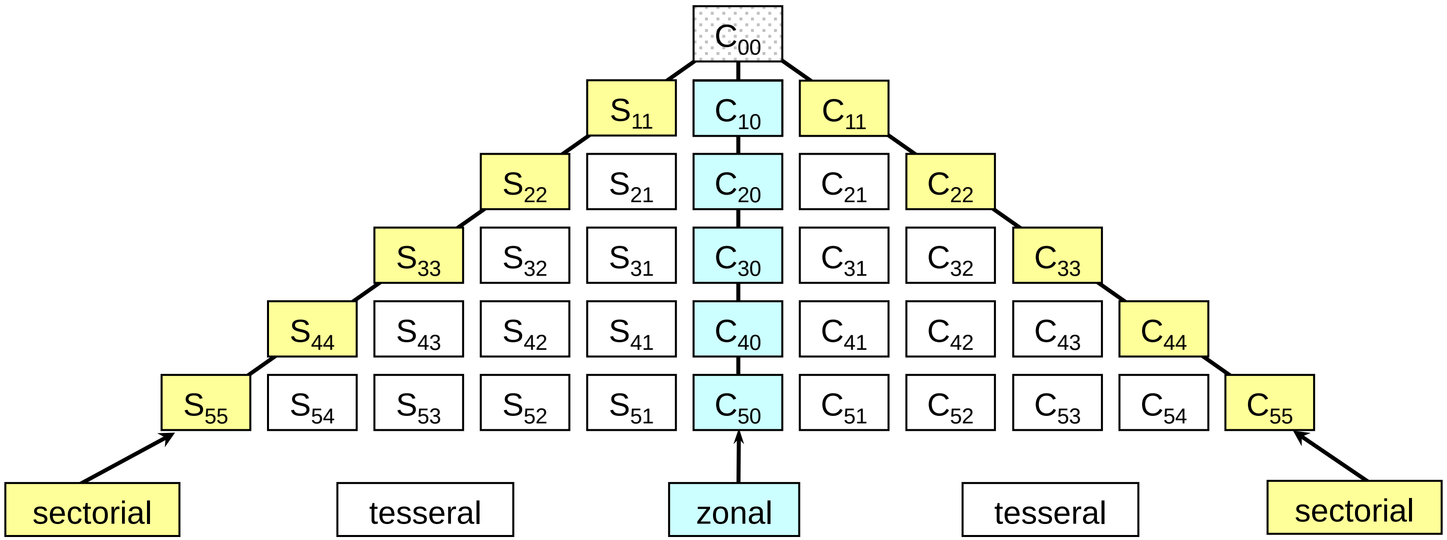
$T = \delta V$

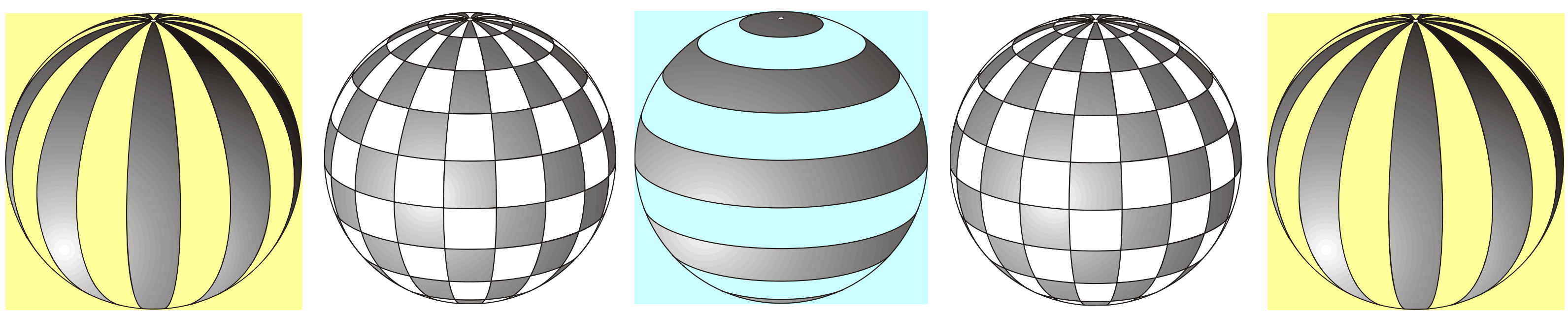
$T_r = \delta V_r$

$T_{rr} = \delta V_{rr}$



series representation of functions on a sphere





surface spherical harmonic functions: $Y_{nm}(\varphi, \lambda) = \bar{P}_{n|m|}(\varphi) \begin{cases} \cos m\lambda \\ \sin m\lambda \end{cases}$