A Relaxed Wishart Model for Polarimetric SAR Data

Stian Normann Anfinsen
Torbjørn Eltoft
Anthony Paul Doulgeris

Department of Physics and Technology
University of Tromsø, Norway
Outline

- Distribution models for PolSAR data
- Goodness-of-Fit Evaluation
- Experiments
- Conclusions
Distribution Models for PolSAR Data

- Complex Wishart distribution – de facto standard
  - Assumes circular complex Gaussian scattering vector.
  - Models fully developed speckle, no texture.
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- Compounded distributions based on product model
  - Accounts for texture. More flexibility, better fit, more complex.
  - E.g.: Matrix-variate $\mathcal{K}$, $\mathcal{G}^0$, and $\mathcal{U}$ distribution.
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  - E.g.: Matrix-variate $\mathcal{K}$, $\mathcal{G}^0$, and $\mathcal{U}$ distribution.

- Proposed alternative: Relaxed Wishart distribution
  - Same functional form as Wishart distribution.
  - $L_e$ treated as free parameter $L$. 
Distribution Models for PolSAR Data

- Multilook polarimetric product model:

\[ C = zW, \quad W \sim \mathcal{W}_d(L_e, \Sigma), \quad z \in \mathbb{R}^+, \quad E\{z\} = 1. \]

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- \( z \sim \Gamma(\alpha) \quad \implies \quad C \sim \mathcal{K}_d(L_e, \Sigma, \alpha) \quad [\text{Lee et al., IGARSS'94}] \)
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- \( z \sim \Gamma^{-1}(\lambda) \quad \implies \quad C \sim \mathcal{G}_d^0(L_e, \Sigma, \lambda) \quad [\text{Freitas et al., Env.'05}] \)
Distribution Models for PolSAR Data

- Complex Wishart distribution:

\[ p_C(C; L_e, \Sigma) = \frac{L_e^{L_e d} |C|^{L_e - d}}{|\Sigma|^{L_e \Gamma_d(L_e)}} \exp \left( -L_e \text{tr} (\Sigma^{-1}C) \right) \]

- \( L_e \): Equivalent Number of Looks (ENL)
  - Global constant
  - Estimated once for whole data set
Distribution Models for PolSAR Data

Matrix-variate $\mathcal{K}$-distribution:

$$p_C(C; L_e, \Sigma, \alpha) = \frac{2|C|^{L_e-d}(L_e\alpha)^{\frac{\alpha+L_e d}{2}}}{|\Sigma|^{L_e}\Gamma_d(L_e)\Gamma(\alpha)} \left( \text{tr}(\Sigma^{-1}C) \right)^{\frac{\alpha-L_e d}{2}} \times \mathcal{K}_{\alpha-L_e d}(2\sqrt{L_e\alpha \text{tr}(\Sigma^{-1}C)})$$

$\alpha$ : texture parameter

$\alpha \to \infty \implies \mathcal{K}_d(L_e, \Sigma, \alpha) \to \mathcal{W}_d(L_e, \Sigma)$
Distribution Models for PolSAR Data

- Matrix-variate $G^0$-distribution:

$$ p_C(C; L_e, \Sigma, \lambda) = \frac{L_e^{Le d} |C|^{Le -d} \Gamma(L_e d + \lambda) (\lambda - 1)^\lambda}{|\Sigma|^{L_e} \Gamma_d(L_e) \Gamma(\lambda)} \times \left( L_e \text{tr}(\Sigma^{-1} C) + \lambda - 1 \right)^{-\lambda - L_e d} $$

- $\lambda$ : texture parameter

- $\lambda \to \infty \quad \Rightarrow \quad G^0_d(L_e, \Sigma, \lambda) \to W_d(L_e, \Sigma)$
Distribution Models for PolSAR Data

Relaxed Wishart distribution:

$$p_C(C; L, \Sigma) = \frac{L^{Ld}|C|^{L-d}}{|\Sigma|^L\Gamma_d(L)} \exp \left( -L \text{ tr} (\Sigma^{-1}C) \right)$$
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- Notation: \( C \sim \mathcal{RW}_d(C; L, \Sigma) \)
Marginal Densities of Intensity

$R_W (L, \Sigma)$

$K_d (L_e, \Sigma, \alpha)$

$G_0 (L_e, \Sigma, \lambda)$
Goodness-of-Fit Evaluation

- Established approach:
  - Visual inspection of marginal densities of intensity
  - Comparison of model densities with data histograms
  - Traditional statistical tests not applicable
Goodness-of-Fit Evaluation

Established approach:

➲ Visual inspection of marginal densities of intensity
➲ Comparison of model densities with data histograms
➲ Traditional statistical tests not applicable

Proposed alternative:

➲ Visual inspection of matrix moments
➲ Comparison of theoretical moments for candidate models with sample moments estimated from data
➲ Hypothesis test under development (IGARSS ’09)
Goodness-of-Fit Evaluation

Conventional matrix moments:

\[
\begin{align*}
E\{\text{tr}(C)\} &= E\{z\} \text{tr}(\Sigma) \\
E\{\text{tr}(CC)\} &= E\{z^2\} \left( \text{tr}(\Sigma\Sigma) + \frac{1}{L} \text{tr}(C)^2 \right)
\end{align*}
\]

Hotelling-Lawley trace moments:

\[
\begin{align*}
E\{\text{tr}(\Sigma^{-1}C)\} &= E\{z\}d \\
\text{Var}\{\text{tr}(\Sigma^{-1}C)\} &= \frac{1}{L} \left( E\{z^2\}d(d + 1) - d^2 \right)
\end{align*}
\]
Goodness-of-Fit Evaluation

Log-determinant cumulants of $C$

- Log-cumulants of single polarisation amplitude/intensity were derived by J.-M. Nicolas (TdS, 2002).
- We extend this theory to matrix-variate statistics and PolSAR.

moments: $m_r(x) \leftrightarrow \phi_x(\omega) = \mathcal{F}\{p_x(x)\}$, for a r.v. $x$

cumulants: $c_r(x) \leftrightarrow \ln \phi_x(\omega)$

log-moments: $\mu_r(x) \leftrightarrow \psi_x(\omega) = \mathcal{M}\{p_x(x)\}$, $x \in \mathbb{R}^+$

log-cumulants: $\kappa_r(x) \leftrightarrow \ln \psi_x(\omega)$
Goodness-of-Fit Evaluation

Log-determinant cumulants of $C$

Plug $x = |C| = |zW| = z^d |W| > 0$ into $\kappa_r(x)$. 
Goodness-of-Fit Evaluation

Log-determinant cumulants of $C$

- Plug $x = |C| = |zW| = z^d |W| > 0$ into $\kappa_r(x)$.
- Mellin transform property $\Rightarrow$

$$\kappa_r(|C|) = d^r \kappa_r(z) + \kappa_r(|W|).$$
Goodness-of-Fit Evaluation

Log-determinant cumulants of $\mathbf{C}$

- Plug $x = |\mathbf{C}| = |z\mathbf{W}| = z^d|\mathbf{W}| > 0$ into $\kappa_r(x)$.
- Mellin transform property $\Rightarrow$

$$
\kappa_r(|\mathbf{C}|) = d^r \kappa_r(z) + \kappa_r(|\mathbf{W}|)
$$

$$
\kappa_{r>1}(|\mathbf{C}|) = \begin{cases} 
  d^r \Psi^{(r-1)}(\alpha) + \sum_{i=0}^{d-1} \Psi^{(r-1)}(L_e - i), & \mathbf{C} \sim \mathcal{K}_d(L_e, \Sigma, \alpha) \\
  (-d)^r \Psi^{(r-1)}(\lambda) + \sum_{i=0}^{d-1} \Psi^{(r-1)}(L_e - i), & \mathbf{C} \sim \mathcal{G}_d^0(L_e, \Sigma, \lambda) \\
  \sum_{i=0}^{d-1} \Psi^{(r-1)}(L - i), & \mathbf{C} \sim \mathcal{RW}_d(L, \Sigma)
\end{cases}
$$
Goodness-of-Fit Evaluation

Log-det cumulant diagram

<table>
<thead>
<tr>
<th>Free Parameters</th>
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<tbody>
<tr>
<td>$\mathcal{W}$</td>
</tr>
<tr>
<td>$\mathcal{K}$</td>
</tr>
<tr>
<td>$\mathcal{G}^0$</td>
</tr>
<tr>
<td>$\mathcal{RW}$</td>
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</tbody>
</table>
Test data: Patches from AIRSAR L-band image of Flevoland, NL (NASA/JPL).
Experiment: Marginal densities for area A

Unfiltered

7 × 7 refined Lee filter
Experiment: Marginal densities for area B

Unfiltered

7 × 7 refined Lee filter
Experiment: Marginal densities for area C

Unfiltered

7 × 7 refined Lee filter
Experiments

Log-determinant cumulant diagrams

Unfiltered

7 × 7 modified Lee filter
Explanation of speckle filter effect

- PDF of unfiltered data:

\( p_c(C; L = L_e, \Sigma) \)

- Assume \( C \) approximately Wishart distributed

- PDF of speckle filtered data

\[
p_c(C; \Sigma, \theta) = \int_{L_e}^{\infty} p_c(C; L, \Sigma) p_L(L; \theta) dL
\]

- Distribution modified by adaptive speckle filtering

- The relaxed Wishart distribution is apparently a good model for the modified distribution
Application to Classification

Distance from pixel $i$ to class $j$:

$$d_{RW}(C_i, \Sigma_j, L_j) = \ln \left( \frac{|\Sigma_j|}{|C_i|} \right) + \frac{1}{L_j} \sum_{k=0}^{d-1} \ln \Gamma(L_j - k) + \text{tr}(\Sigma_j^{-1} C_i) - d \ln(L_j)$$

Compare with Wishart classifier:

$$d_{W}(C_i, \Sigma_j) = \ln |\Sigma_j| + \text{tr}(\Sigma_j^{-1} C_i)$$
Conclusions

- We have proposed the relaxed Wishart $\mathcal{RW}$ distribution with a non-constant shape parameter.
- We have shown that log-determinant cumulants of the polarimetric coherency/covariance matrix can be used in visual inspection of goodness-of-fit.
- Experimental results demonstrate that the $\mathcal{RW}$ distribution can compete with distributions derived from the product model, especially for modelling of speckle filtered data.
- The $\mathcal{RW}$ distribution should be tested in model-based classification and change detection, and other image analysis tasks.