A Relaxed Wishart Model for Polarimetric SAR Data

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Outline

- Distribution models for PolSAR data
- Goodness-of-Fit Evaluation
- Experiments
- Conclusions

Distribution Models for PolSAR Data

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  - Assumes circular complex Gaussian scattering vector.
  - Models fully developed speckle, no texture.

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  - Accounts for texture. More flexability, better fit, more complex.
  - E.g.: Matrix-variate $\mathcal{K}$, $\mathcal{G}^0$, and $\mathcal{U}$ distribution.
- Proposed alternative: Relaxed Wishart distribution
  - Same functional form as Wishart distribution.
  - L_e treated as free parameter L.

Distribution Models for PolSAR Data

- Multilook polarimetric product model:

$$\mathbf{C} = z\mathbf{W}\,, \quad \mathbf{W} \sim \mathcal{W}_d(L_e, \boldsymbol{\Sigma})\,,\; z \in \mathbb{R}^+\,,\; \mathrm{E}\{z\} = 1\,.$$

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- $z \sim \Gamma(\alpha) \quad \Longrightarrow \quad \mathbf{C} \sim \mathcal{K}_d(L_e, \boldsymbol{\Sigma}, \alpha)$ [Lee et al., IGARSS'94]
- $z \sim \Gamma^{-1}(\lambda) \quad \Longrightarrow \quad \mathbf{C} \sim \mathcal{G}_d^0(L_e, \boldsymbol{\Sigma}, \lambda)$ [Freitas et al., Env.'05]

Distribution Models for PolSAR Data

- Complex Wishart distribution:

$$p_{\mathbf{C}}(\mathbf{C}; L_e, \boldsymbol{\Sigma}) = \frac{L_e^{L_e d} |\mathbf{C}|^{L_e - d}}{|\boldsymbol{\Sigma}|^{L_e} \Gamma_d(L_e)} \exp\left(-L_e \operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{C}\right)\right)$$

- L_e : Equivalent Number of Looks (ENL)
  - Global constant
  - Estimated once for whole data set

Distribution Models for PolSAR Data

- Matrix-variate $\mathcal{K}$-distribution:

$$p_C(\mathbf{C}; L_e, \mathbf{\Sigma}, \alpha) = \frac{2|\mathbf{C}|^{L_e-d}(L_e\alpha)^{\frac{\alpha+L_e d}{2}}}{|\mathbf{\Sigma}|^{L_e}\Gamma_d(L_e)\Gamma(\alpha)}\left(\operatorname{tr}(\mathbf{\Sigma}^{-1}\mathbf{C})\right)^{\frac{\alpha-L_e d}{2}} \times K_{\alpha-L_e d}\left(2\sqrt{L_e\alpha\operatorname{tr}(\mathbf{\Sigma}^{-1}\mathbf{C})}\right)$$

- α : texture parameter
- $\alpha \to \infty \quad \Longrightarrow \quad \mathcal{K}_d(L_e, \mathbf{\Sigma}, \alpha) \to \mathcal{W}_d(L_e, \mathbf{\Sigma})$

Distribution Models for PolSAR Data

- Matrix-variate $\mathcal{G}^0$-distribution:

$$p_{\mathbf{C}}(\mathbf{C}; L_e, \boldsymbol{\Sigma}, \lambda) = \frac{L_e^{L_e d} |\mathbf{C}|^{L_e - d}}{|\boldsymbol{\Sigma}|^{L_e}} \frac{\Gamma(L_e d + \lambda)(\lambda - 1)^{\lambda}}{\Gamma_d(L_e)\Gamma(\lambda)} \times \left(L_e \operatorname{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{C}) + \lambda - 1\right)^{-\lambda - L_e d}$$

- λ : texture parameter
- $\lambda \to \infty \quad \Longrightarrow \quad \mathcal{G}_d^0(L_e, \boldsymbol{\Sigma}, \lambda) \to \mathcal{W}_d(L_e, \boldsymbol{\Sigma})$

Distribution Models for PolSAR Data

- Relaxed Wishart distribution:

$$p_{\mathbf{C}}(\mathbf{C}; L, \boldsymbol{\Sigma}) = \frac{L^{Ld} |\mathbf{C}|^{L-d}}{|\boldsymbol{\Sigma}|^L \Gamma_d(L)} \exp\left(-L \operatorname{tr}\left(\boldsymbol{\Sigma}^{-1}\mathbf{C}\right)\right)$$

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  - Varies between classes, segments, or even pixels
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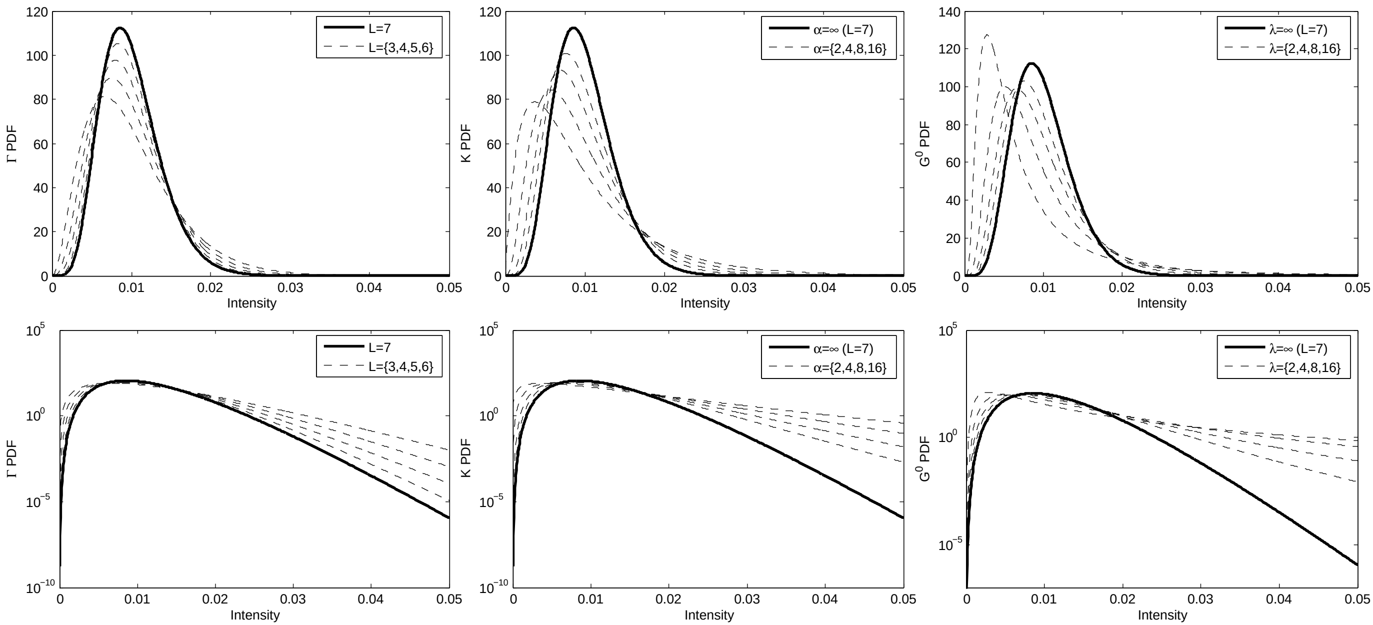
- Notation: $\mathbf{C} \sim \mathcal{RW}_d(\mathbf{C}; L, \boldsymbol{\Sigma})$

Marginal Densities of Intensity

$\mathcal{RW}_d(L, \mathbf{\Sigma})$ $\mathcal{K}_d(L_e, \mathbf{\Sigma}, \alpha)$ $\mathcal{G}^0_d(L_e, \mathbf{\Sigma}, \lambda)$



Goodness-of-Fit Evaluation

- Established approach:
  - Visual inspection of marginal densities of intensity
  - Comparison of model densities with data histograms
  - Traditional statistical tests not applicable

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- Proposed alternative:
  - Visual inspection of *matrix moments*
  - Comparison of theoretical moments for candidate models with sample moments estimated from data
  - Hypothesis test under development (IGARSS '09)

Goodness-of-Fit Evaluation

Conventional matrix moments:

$$\begin{aligned} \mathrm{E}\{\mathrm{tr}(\mathbf{C})\} &= \mathrm{E}\{z\}\,\mathrm{tr}(\mathbf{\Sigma}) \\ \mathrm{E}\{\mathrm{tr}(\mathbf{CC})\} &= \mathrm{E}\{z^2\}\left(\mathrm{tr}(\mathbf{\Sigma\Sigma}) + \frac{1}{L}\,\mathrm{tr}(\mathbf{C})^2\right) \end{aligned}$$

Hotelling-Lawley trace moments:

$$\begin{aligned} \mathrm{E}\{\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{C})\} &= \mathrm{E}\{z\}d \\ \mathrm{Var}\{\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{C})\} &= \frac{1}{L}\left(\mathrm{E}\{z^2\}d(d+1) - d^2\right) \end{aligned}$$

Goodness-of-Fit Evaluation

Log-determinant cumulants of $\mathbf{C}$

- Log-cumulants of single polarisation amplitude/intensity were derived by J.-M. Nicolas (TdS, 2002).
- We extend this theory to matrix-variate statistics and PolSAR.

$$
\begin{array}{lll}
\text{moments:} & m_r(x) \leftrightarrow & \phi_x(\omega) = \mathfrak{F}\{p_x(x)\}\,, \quad \text{for a r.v. } x \\
\text{cumulants:} & c_r(x) \leftrightarrow & \ln \phi_x(\omega) \\
\text{log-moments:} & \mu_r(x) \leftrightarrow & \psi_x(\omega) = \mathfrak{M}\{p_x(x)\}\,, \quad x \in \mathbb{R}^+ \\
\text{log-cumulants:} & \kappa_r(x) \leftrightarrow & \ln \psi_x(\omega)
\end{array}
$$

Goodness-of-Fit Evaluation

Log-determinant cumulants of $\mathbf{C}$

- Plug $x = |\mathbf{C}| = |z\mathbf{W}| = z^d|\mathbf{W}| > 0$ into $\kappa_r(x)$.

Goodness-of-Fit Evaluation

Log-determinant cumulants of $\mathbf{C}$

- Plug $x = |\mathbf{C}| = |z\mathbf{W}| = z^d|\mathbf{W}| > 0$ into $\kappa_r(x)$.
- Mellin transform property $\Rightarrow$

$$\kappa_r(|\mathbf{C}|) = d^r\kappa_r(z) + \kappa_r(|\mathbf{W}|)\,.$$

Goodness-of-Fit Evaluation

Log-determinant cumulants of **C**

- Plug $x = |\mathbf{C}| = |z\mathbf{W}| = z^d|\mathbf{W}| > 0$ into $\kappa_r(x)$.
- Mellin transform property $\Rightarrow$

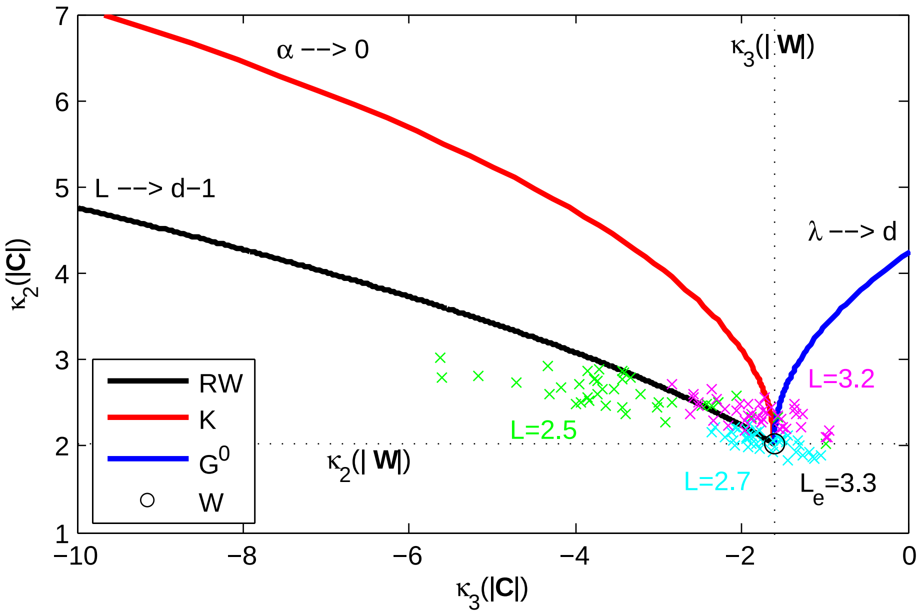
$$\kappa_r(|\mathbf{C}|) = d^r \kappa_r(z) + \kappa_r(|\mathbf{W}|)$$

$$\kappa_{r>1}(|\mathbf{C}|) = \begin{cases} d^r \Psi^{(r-1)}(\alpha) + \sum_{i=0}^{d-1} \Psi^{(r-1)}(L_e - i)\,, & \mathbf{C} \sim \mathcal{K}_d(L_e, \boldsymbol{\Sigma}, \alpha) \\ (-d)^r \Psi^{(r-1)}(\lambda) + \sum_{i=0}^{d-1} \Psi^{(r-1)}(L_e - i)\,, & \mathbf{C} \sim \mathcal{G}_d^0(L_e, \boldsymbol{\Sigma}, \lambda) \\ \sum_{i=0}^{d-1} \Psi^{(r-1)}(L - i)\,, & \mathbf{C} \sim \mathcal{RW}_d(L, \boldsymbol{\Sigma}) \end{cases}$$

Goodness-of-Fit Evaluation

Log-det cumulant diagram

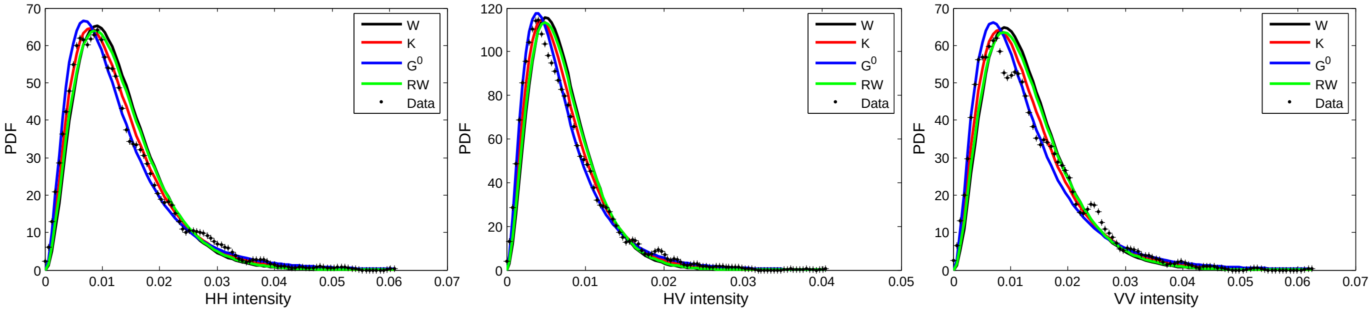
Free Parameters	
$\mathcal{W}$	None
$\mathcal{K}$	α
$\mathcal{G}^0$	λ
$\mathcal{RW}$	L



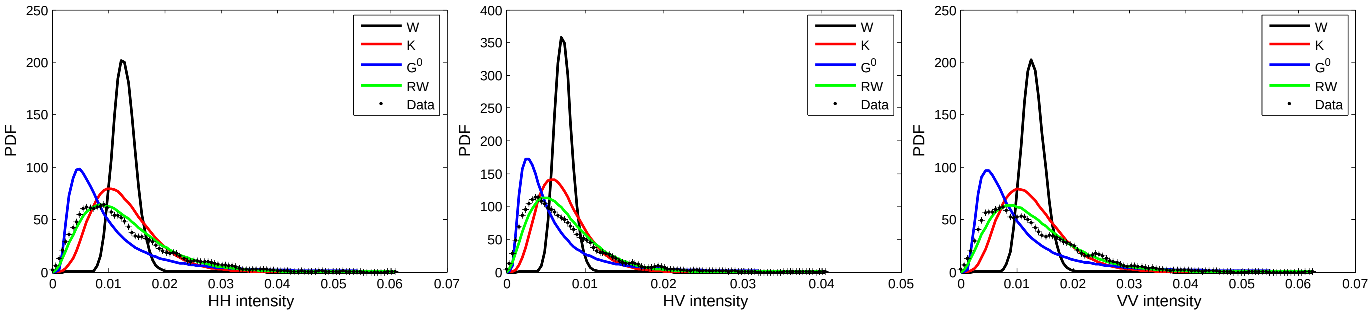
Test data: Patches from AIRSAR L-band image of Flevoland, NL (NASA/JPL).

Experiment: Marginal densities for area A

Unfiltered

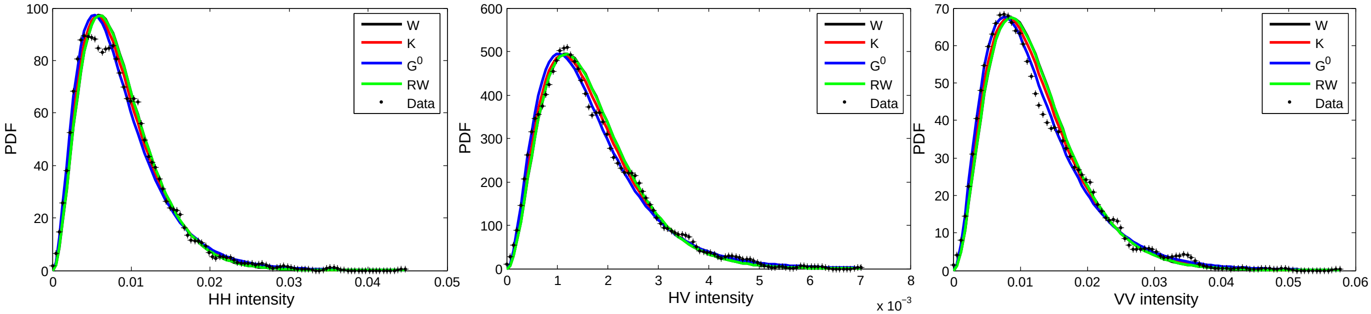


7×7 refined Lee filter

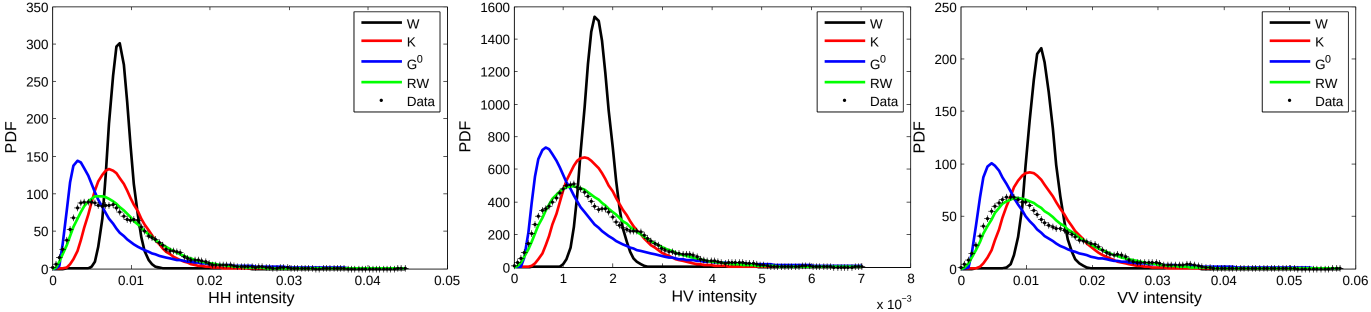


Experiment: Marginal densities for area B

Unfiltered

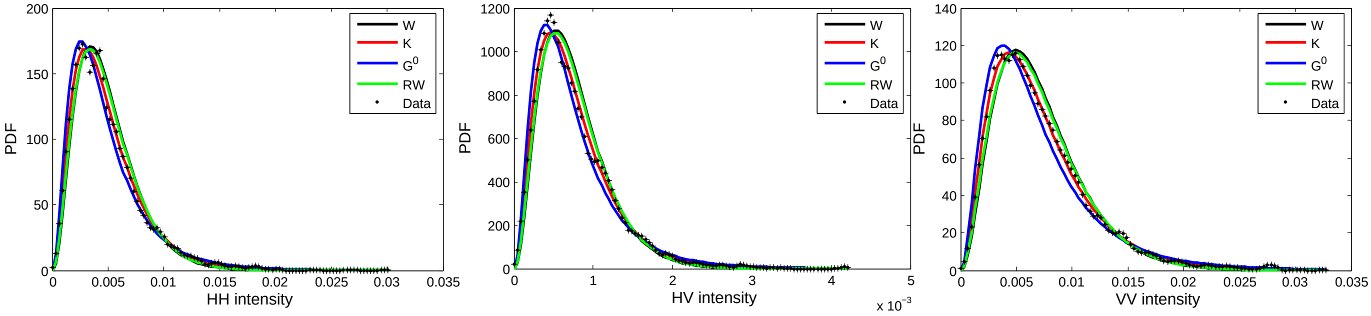


7×7 refined Lee filter

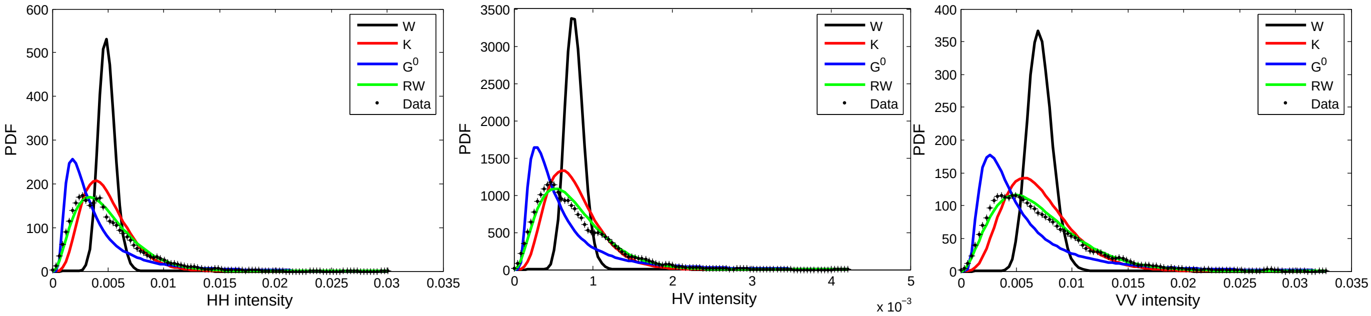


Experiment: Marginal densities for area C

Unfiltered



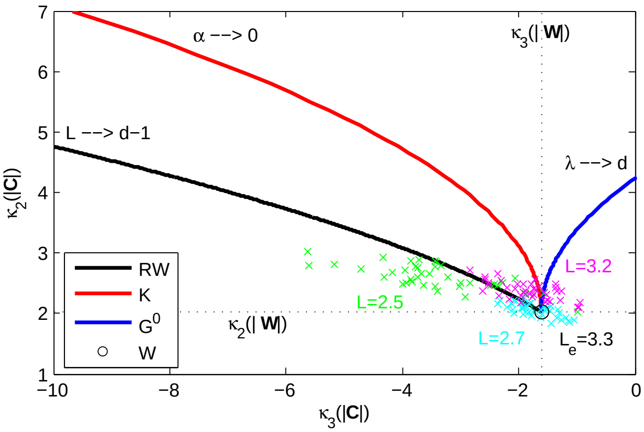
7 × 7 refined Lee filter



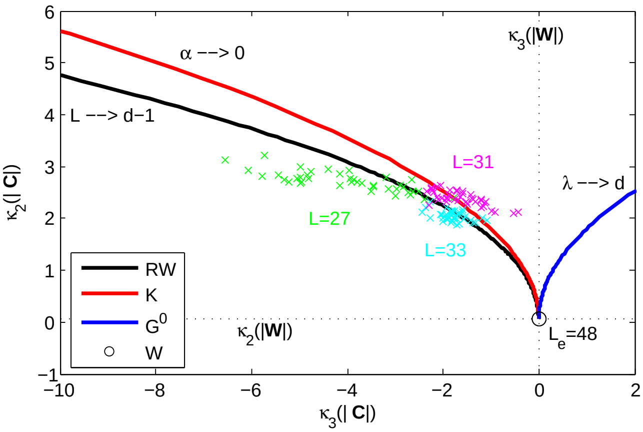
Experiments

Log-determinant cumulant diagrams

Unfiltered



7×7 modified Lee filter



Explanation of speckle filter effect

- PDF of unfiltered data:

$$p_{\mathbf{C}}(\mathbf{C}; L = L_e, \boldsymbol{\Sigma})$$

  - Assume **C** approximately Wishart distributed
- PDF of speckle filtered data

$$p_{\mathbf{C}}(\mathbf{C}; \boldsymbol{\Sigma}, \boldsymbol{\theta}) = \int_{L_e}^{\infty} p_{\mathbf{C}}(\mathbf{C}; L, \boldsymbol{\Sigma}) p_L(L; \boldsymbol{\theta})\, dL$$

  - Distribution modified by adaptive speckle filtering
  - The relaxed Wishart distribution is apparently a good model for the modified distribution

Application to Classification

- Distance from pixel i to class j:

$$d_{\mathcal{RW}}(\mathbf{C}_i, \mathbf{\Sigma}_j, L_j) = \ln\left(\frac{|\mathbf{\Sigma}_j|}{|\mathbf{C}_i|}\right) + \frac{1}{L_j}\sum_{k=0}^{d-1} \ln\Gamma(L_j - k) + \mathrm{tr}(\mathbf{\Sigma}_j^{-1}\mathbf{C}_i) - d\ln(L_j)$$

- Compare with Wishart classifier:

$$d_{\mathcal{W}}(\mathbf{C}_i, \mathbf{\Sigma}_j) = \ln|\mathbf{\Sigma}_j| + \mathrm{tr}(\mathbf{\Sigma}_j^{-1}\mathbf{C}_i)$$

Conclusions

- We have proposed the relaxed Wishart $\mathcal{RW}$ distribution with a non-constant shape parameter.
- We have shown that log-determinant cumulants of the polarimetric coherency/covariance matrix can be used in visual inspection of goodness-of-fit.
- Experimental results demonstrate that the $\mathcal{RW}$ distribution can compete with distributions derived from the product model, especially for modelling of speckle filtered data.
- The $\mathcal{RW}$ distribution should be tested in model-based classification and change detection, and other image analysis tasks.