COMPARISON OF SOME METHODS FOR MODIFYING
STOKES' FORMULA IN THE GOCE ERA

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ABSTRACT

The dedicated satellite gravity mission GOCE will drastically improve our knowledge of the long to medium wavelengths of the Earth's gravity field. In order to determine the finest details in regional geoid determination, however, we still have to utilise gravity data. It is the purpose of this paper to study three modifications of Stokes' formula numerically, using error propagations with simulated standard errors for the GOCE potential coefficients. The methods tested are the standard remove-compute-restore, the least squares, and the low-degree GOCE-only modifications. In the latter technique it is required that only GOCE information must influence the determination of the lowest degrees. It is concluded that of the modifications tested, the least squares method is most suitable to be used with a GOCE satellite-only model. This is the case also when pessimistic weights are used for the gravity anomalies. The main fault with the standard remove-compute-restore method is its sensitivity to long-wavelength errors in the gravity anomalies, while a very large truncation error is the most serious problem for the low-degree GOCE-only technique.

1 INTRODUCTION

The GOCE satellite, to be launched during 2006, will utilise Satellite Gravity Gradiometry (SGG) and Satellite to Satellite Tracking in the high-low (SST-hl) mode to determine the long to medium wavelengths of the Earth's gravity field. According to the performed simulations (e.g. [5] and [1]), the geoid commission error is expected to be somewhere around 1 cm for a resolution of 80 km (corresponding to the spherical harmonic maximum degree \(M = 200\)). Even though this performance is truly astonishing, terrestrial gravity will still be needed to determine the highest harmonics in a regional geoid determination. Now, terrestrial gravity observations are usually distributed in inhomogeneous ways and they are often affected by systematic errors. As such errors are difficult to detect, there is always the risk that the high accuracy obtained from GOCE for the lower degrees is ruined in the combined solution. Furthermore, if it is assumed that the GOCE Earth Gravity Model (EGM) is used up to the degree \(M\), for which a geoid commission RMS-error of 1 cm or so is obtained, it is not reasonable to believe that the EGM can be improved by terrestrial gravity anomalies. It is therefore crucial to choose a kernel modification that is insensitive to the long-wavelength part of the terrestrial gravity anomaly spectrum. As the maximum degree \(M\) is not too high, the size of the truncation error is also an important factor to be considered in the GOCE case. In order to avoid long-wavelength errors in the gravity anomalies, [11] proposed the low-degree GOCE-only modification. This technique is derived to filter out the gravity anomaly power below degree \(M+1\) completely. Another method that might be suitable is the least squares modification [8]. It is the main purpose of this paper to study the error propagation of these kernel modifications and compare them with the standard remove-compute-restore (r-c-r) method that utilises an unmodified kernel (see e.g. [7]). The focus is mainly on the aspects that are important when a kernel modification method is used together with a GOCE satellite-only EGM. It is noted that also other modifications might be suitable, for instance the Wong and Gore method [15], but these investigations are left for the future.

2 THE UNBIASED GEOID ESTIMATOR

It is assumed that the geoid is computed by means of a remove-compute-restore type of estimator using some modification of Stokes' function. Both the terrestrial gravity anomalies as well as the EGM contributions are corrected for the direct topographic effect, and the indirect topographic effect is finally added to the geoid height; see e.g. [3] and [9] for details. The topographic reduction method is left unspecified. In order to keep the formulas below as simple as possible, no explicit corrections are introduced for the zero and first degree topographic effects [10]. Atmospheric and ellipsoidal corrections are also implicitly understood. Now, if it is assumed that only gravity anomalies over a spherical cap \(\sigma_0\) with radius \(\psi_0\) are utilised, the estimator becomes:
\[
\dot{N} = \frac{1}{2\pi} \int S_{k}(\psi) \left( \Delta g - \Delta g_{\text{OGM}}^{\text{DR}} + \sum_{n=2}^{M} \left( \frac{R}{n^2} \right)^{n-2} \left( \Delta g_{\text{EGM}}^{\text{DR}} - \Delta g_{\text{OGM}}^{\text{DR}} \right) \right) \cdot d\sigma + \sum_{n=2}^{M} \frac{2}{n^2} \left( \Delta g_{\text{EGM}}^{\text{DR}} - \Delta g_{\text{OGM}}^{\text{DR}} + \delta N_{i} \right)
\]

(1)

where \( \Delta g \) is the observed terrestrial gravity anomaly at the Earth’s surface, \( \Delta g_{\text{EGM}}^{\text{EGM}} \) is the gravity anomaly Laplace harmonic of degree \( n \) for the EGM and \( c = R/(2\gamma) \). Here \( R \) is the mean Earth radius and \( \gamma \) is the mean normal gravity at sea level. Furthermore, the direct topographic effect on the surface gravity anomaly is denoted by \( \Delta g_{\text{DR}}^{\text{DR}} \), while the indirect effect on the geoid height is given by \( \delta N_{i} \). A star ‘*’ is used to denote downward continuation to sea level. The modified Stokes’ function is defined as

\[
S_{k}(\psi) = S(\psi) - \sum_{k=2}^{L} \frac{k+1}{2} s_{k} P_{k}(\cos \psi)
\]

(2)

where \( S(\psi) \) is the original Stokes’ function, \( P_{k}(\cos \psi) \) is the Legendre polynomial of degree \( k \) and \( s_{k} \) are the modification parameters. The maximum degree of modification \( L \) is taken to be equal to or larger than the maximum degree \( M \) of the EGM. We now introduce \( \epsilon_{\text{OGM}}^{\text{DR}} \) and \( \epsilon_{\text{EGM}}^{\text{DR}} \) for the random errors in the topographically corrected Laplace harmonics of the downward continued gravity anomaly and the EGM, respectively. The errors are assumed to be uncorrelated and to have zero expectation for all \( n \). It is shown in [12] that the spectral form of Eq. (1) is given by

\[
\dot{N} = \sum_{n=2}^{M} \left( \frac{2}{n-1} - s'_{n} - Q_{n}^{L} \right) \left( \Delta g_{n} - \Delta g_{\text{OGM}}^{\text{DR}} + \epsilon_{n}^{\text{DR}} \right) + \sum_{n=2}^{M} \left( Q_{n}^{L} + s_{n} \right) \left( \Delta g_{n} - \Delta g_{\text{OGM}}^{\text{DR}} + \epsilon_{n}^{\text{EGM}} \right) + \delta N_{i}
\]

(3)

where \( s'_{n} = s_{n} \) whenever \( n \leq L \) and \( s'_{n} = 0 \) otherwise. The truncation coefficients for the modified Stokes’ kernel \( Q_{n}^{L} \) are defined as

\[
Q_{n}^{L} = \sum_{s_{n}} S_{k}(\psi) P_{k}(\cos \psi) \sin \psi d\psi = Q_{n} - \sum_{k=2}^{L} \frac{k+1}{2} s_{k} e_{\psi}
\]

(4)

in which \( Q_{n} \) are the truncation coefficients for the original Stokes’ function and \( e_{\psi} \) are the so-called Paul’s coefficients. Neglecting the harmonics of degree zero and one, the true geoid height becomes

\[
\dot{N} = \sum_{n=2}^{M} \frac{2}{n-1} \left( \Delta g_{n} - \Delta g_{\text{OGM}}^{\text{DR}} \right) + \delta N_{i}
\]

(5)

It is now assumed that the error covariance functions for the (topographically corrected) terrestrial gravity and the EGM are homogeneous and isotropic. The corresponding error degree variances are denoted by \( \sigma_{\text{OGM}}^{2} \) and \( \sigma_{\text{EGM}}^{2} \). Using \( c_{n} \) for the (signal) gravity anomaly degree variances of the topographically reduced field, the expected global mean square error becomes [8],

\[
\delta \dot{N}^{2} = \frac{1}{4\pi} \iint \left( \dot{N} - N \right)^{2} d\sigma = c^{2} \sum_{n=2}^{M} \left( \frac{2}{n-1} - s_{n} - Q_{n}^{L} \right)^{2} \sigma_{\text{OGM}}^{2} + \left( s_{n} + Q_{n}^{L} \right)^{2} \sigma_{\text{EGM}}^{2}
\]

(6)

+ \sum_{n=L+1}^{M} \left( \frac{2}{n-1} - s_{n} - Q_{n}^{L} \right)^{2} \sigma_{\text{OGM}}^{2} + \left( s_{n} + Q_{n}^{L} \right)^{2} \sigma_{\text{EGM}}^{2} + \sum_{n=L+1}^{M} \left( \frac{2}{n-1} - Q_{n}^{L} \right)^{2} c_{n}^{2}

Eq. (6) is fundamental in this paper as it is used to compare different modification methods. If the parameters \( s_{n} \) are given, the expected global RMS-error can be computed for the specified values of the signal and noise degree variances. The terms containing \( \sigma_{\text{OGM}}^{2} \) and \( \sigma_{\text{EGM}}^{2} \) are the contributions from the errors in the terrestrial gravity data and the EGM, respectively, while those parts containing \( c_{n} \) represent the bias or truncation error. It is also possible to study the contribution for different frequency bands. Note that no bias is present for the degrees up to \( M \), which is the reason for [8] calling the estimator in Eq. (1) unbiased.
3 SIGNAL AND NOISE DEGREE VARIANCES

In this section the signal and error degree variance models that were utilised in the error propagation studies below are presented. The signal degree variance model used for the topographically reduced field above degree $M$ is of Tscherning and Rapp type [14]:

$$c_n = A \left( \frac{n-1}{n-2} \right) \left( \frac{R_g}{R} \right)^{n-2}$$

with the parameters chosen to $R = 6371$ km, $A = 225$ mGal$^2$, $B = 4$, and $R - R_g = 3.5$ km as specified in Forsberg [4]. The degree variances obtained by Eq. (7) agree well on average with the spectral investigations of the topographically corrected gravity field in [3] and [4].

The shape of the noise degree variance curve for the EGM stemming from GOCE was taken from Fig. 4.1 in [5]. The scale was chosen so that a global commission RMS-error of 1 cm is obtained for the maximum degree $M = 200$. This is a little bit more pessimistic than the value in Table 8.4 in [5], but is in good agreement with other simulations (e.g. [1]). However, for the present study, the scale of the GOCE noise degree variance model is not very crucial. What is important is that the “shape” is reasonable at the same time as the commission error is practically negligible up to some rather high maximum degree $M$; cf. the introduction. These requirements are fulfilled by the chosen model, which is illustrated in Fig. 1. It is noted that the quality in the GOCE polar gaps are likely to be worse compared to the rest of the world [13], of course depending on how and to what extent the gaps are filled. This means that the use of a homogeneous and isotropic error model is not strictly justified. However, we may view our “global” error degree variances as representative of the situation outside the polar gaps only, where the error field is almost homogeneous and isotropic [5].

![Fig. 1a,b. Degree-order standard deviations for GOCE and the two gravity anomaly noise models with $\sigma = 1$ mGal, illustrated using different scales on the horizontal and vertical axes.](image)

It is difficult to choose a representative error model for the terrestrial gravity anomalies. Since their quality depends very much on location, the use of a homogeneous and isotropic covariance function is questionable. However, in the same way as for GOCE above, a “global” error model need only be viewed as representative of the area over which it is used. This means that we may view the error degree variances as describing a global field with the same quality as is exemplified by the gravity anomalies inside the spherical cap in Eq. (1). It should be further noticed that the size of the spherical cap limits how much different frequencies are affected by the gravity anomaly errors, but this fact should not be reflected in the choice of error degree variances. It is automatically taken care of by the truncation coefficients in Eq. (6). In the simulations below two different noise models were utilised, which both have the variance $\sigma^2$ distributed below the Nyquist degree (frequency) of the gravity anomaly grid. Above this degree, the signal degree variance model in Eq. (7) was used to model the presence of high frequency interpolation errors. The first error model assumes that the observation noise is uncorrelated, which is approximately modelled by bandlimited white noise with constant degree-order variance, $\sigma^2_{n,\text{dep}}/(2n+1)$, up to the Nyquist degree; see e.g. [6]. As can be seen in Figs. 1a and 1b, the resulting degree variances for $\sigma = 1$ mGal seem rather optimistic compared to the GOCE-model. In order to model the situation...
when systematic errors are present, the second error model describes correlated noise, which contains more power in the lower portion of the spectrum. It is constructed as a mix of uncorrelated and correlated noise, each containing half the total variance, i.e. $\sigma^2/2$. The uncorrelated part is modelled as bandlimited white noise in the same way as above, but with half the variance, while the correlated part is quite arbitrarily defined with the shape indicated in Figs. 1a and 1b, so that it contains the other half of the variance. The reasons behind choosing this model are that the degree-order standard deviations should be realistic in comparison to GOCE and that the terrestrial data should be affected by long-wavelength systematic errors that decrease with the degree.

In all error propagations presented below, the signal and noise degree variance models presented above were utilised. The gravity anomalies were assumed to be given in an equiangular 3’x3’ grid, which corresponds to the Nyquist degree 3600. The spherical cap was taken to have the radius $\psi_o = 5^\circ$. In order to limit the influence of long-wavelength errors in the gravity data, the cap should not be too large, while a too small radius yields a large truncation error. The chosen value seems like a reasonable compromise. Furthermore, the standard deviation of the gravity anomaly noise was taken as $\sigma = 1$ mGal, and it was assumed that either the correlated or uncorrelated noise model constituted the truth. It should finally be mentioned that the summations to infinity in Eq. (6) were summed to degree 10800 in practice.

### 4 THE STANDARD REMOVE-COMPUTE-RESTORE METHOD

The first modification method to be tested was the standard remove-compute-restore (r-c-r) method (see e.g. [7]), in which Stokes’ function is not modified at all. All modification parameters are consequently set to zero. The corresponding expected global RMS-errors were computed by Eq. (6) using both the uncorrelated and the correlated gravity anomaly error models. The results are presented in Table 1. As noted at the end of Sect. 2, RMS-errors are presented also for different frequency bands and for the parts corresponding to the influence of errors in the gravity anomalies, to the errors in the EGM and to the truncation error (or bias).

<table>
<thead>
<tr>
<th>Noise model for $\Delta g$</th>
<th>Degrees</th>
<th>Uncorrelated, $\sigma = 1$ mGal</th>
<th>Correlated, $\sigma = 1$ mGal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total</td>
<td>All 2 - $M$ M+1 - $\infty$</td>
<td>All 2 - $M$ M+1 - $\infty$</td>
</tr>
<tr>
<td>Global RMS [mm]</td>
<td>EGM (GOCE)</td>
<td>1.6</td>
<td>-</td>
</tr>
<tr>
<td>Bias</td>
<td>40.5</td>
<td>40.5</td>
<td>-</td>
</tr>
</tbody>
</table>

By studying Table 1 it can be seen that the standard r-c-r method is very sensitive to the presence of correlations in the gravity anomaly data. The RMS contribution is large for the degrees up to $M = 200$, while the situation improves drastically when the observations are uncorrelated. As it is believed that the correlated noise model is not totally unrealistic in many cases, considering the well-known fact that gravity data are often affected by systematic errors, it is concluded that the simple r-c-r method is not suitable for use together with a high-accuracy GOCE-model. It is too sensitive to long-wavelength errors in the terrestrial gravity data. Another problem with the standard r-c-r method is the significant truncation error.

### 5 THE LOW-DEGREE GOCE-ONLY MODIFICATION

In order to block the influence of long-wavelength errors in the gravity anomaly data, [11] proposed the low-degree GOCE-only method, which implies that the modification parameters $s_e$ are chosen so that the gravity anomaly contribution is exactly zero up to the maximum degree $M$ of the GOCE-derived EGM. It follows from Eqs. (4) and (6), that this situation is achieved when the following low-degree GOCE-only conditions are fulfilled:

$$s_e = \sum_{n=2}^{M} \frac{2k+1}{2} c_{nk} s_n = \frac{2}{n-1} Q_e; \quad n = 2, 3, \ldots, M$$

In case $M = L$, Eq. (8) provides us with a system of equations with the same number of equations as unknowns. However, due to the fact that the unbiased estimator in Eq. (1) yields numerically the same geoid height for all choices
of $s_i$ for which the modified kernels coincide inside the spherical cap $\sigma_0$ (no matter how the kernels behaves outside the cap), standard truncated singular value decomposition, or some other numerical technique, needs to be applied to find a solution. As Eq. (8) contains the only conditions that are posed on the parameters, the method in question will be called the pure low-degree GOCE-only modification in this paper. A hybrid version of the method for the case $L > M$ is also suggested in [11]. In this case, the modification parameters are chosen so that the expected global mean square error in Eq. (6) is minimised, at the same time as the M conditions in Eq. (8) are fulfilled. This method is called the optimum low-degree GOCE-only modification and requires apriori knowledge of the signal and gravity anomaly error degree variances in Eq. (6). The optimum solution can easily be found using well-known methods from the differential calculus, utilising Lagrange multipliers to ensure the fulfilment of the low-degree GOCE-only condition. The solution is obtained by solving the resulting system of equations by truncated singular value decomposition. Now, using the same input data and type of table as before, but limiting us to the correlated noise model for the gravity anomalies, the propagated RMS-values for the pure and optimum low-degree GOCE-only methods are presented in Table 2. The same signal and noise degree variances as applied in the error propagation were used as apriori values for the weighting.

Table 2. Expected global RMS-errors for the pure and optimum low-degree GOCE-only modification methods. Correlated noise with $\sigma = 1 \text{ mGal}$ used as error model for $\Delta g$.

<table>
<thead>
<tr>
<th>Method, Parameters</th>
<th>Degrees</th>
<th>Pure, $M = L = 200$</th>
<th>Optimum, $M = 200, L = 720$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>2 - $M$</td>
<td>$M+1 - \infty$</td>
</tr>
<tr>
<td>Global RMS [mm]</td>
<td>All</td>
<td>214.6 10.0 214.4 213.4 10.0 213.1 5.1</td>
<td></td>
</tr>
<tr>
<td>EGM (GOCE) Bias</td>
<td>7.1</td>
<td>10.0 0.0 7.1 7.2 0.0 6.6 3.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10.0</td>
<td>10.0 - 10.0 10.0 - -</td>
<td></td>
</tr>
<tr>
<td></td>
<td>214.3</td>
<td>- 214.3 213.1 - 213.0 4.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>11.6</td>
<td>5.4 10.3 10.8 3.5 10.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.9</td>
<td>3.9 - 5.8 5.8 - -</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>1.5 8.6 8.6 - 8.6</td>
<td></td>
</tr>
</tbody>
</table>
| As can be seen in Table 2, the gravity anomaly errors do not influence the degrees up to $M$, which means that the GOCE-model is left completely untouched exactly as required. On the negative side, we see that the basic problem is the huge truncation error, which does not reduce significantly as additional parameters are added in the optimum version of the method. It seems impossible to get rid of the truncation error as long as the GOCE-only condition in Eq. (8) must be strictly fulfilled. Hence, we conclude that the pure and optimum low-degree GOCE-only methods are not good ideas.

6 THE LEAST SQUARES MODIFICATION METHOD

Considering the results in the last section, it seems best to allow for some low-degree influence of gravity anomaly errors, as long as the expected global RMS-error in Eq. (6) is small. Thus, we are led to the least squares modification of Stokes’ formula [8], in which the modification parameters are chosen so that the total expected global mean square error (Eq. 6) of the unbiased geoid estimator is minimised. As in Sect. 5, the solution is provided by the methods of the differential calculus. This gives us a system of equations [8], which can be solved by truncated singular value decomposition. It is also possible to utilise other numerical methods; see [2]. The method naturally requires apriori values for the signal and noise degree variances. Assuming that the correct apriori models are known, the propagated RMS-errors for the correlated error model are presented in the left part of Table 3.

Table 3. Expected global RMS-errors for the least squares modification method using true and pessimistic apriori noise model for $\Delta g$. $M = L = 200$.

<table>
<thead>
<tr>
<th>True noise model $\Delta g$</th>
<th>Correlated, $\sigma = 1 \text{ mGal}$</th>
<th>Correlated, $\sigma = 1 \text{ mGal}$</th>
<th>Uncorrelated, $\sigma = 20 \text{ mGal}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degrees</td>
<td>All 2 - $M$ $M+1 - \infty$</td>
<td>All 2 - $M$ $M+1 - L$ $L+1 - \infty$</td>
<td>All 2 - $M$ $M+1 - \infty$</td>
</tr>
<tr>
<td>Global RMS [mm]</td>
<td>Total 12.3 6.7 10.4 14.9 6.8 13.3</td>
<td>10.8 3.5 10.2</td>
<td></td>
</tr>
<tr>
<td>EGM (GOCE) Bias</td>
<td>11.6 5.4 10.3 5.8 5.8 -</td>
<td>8.6 - 8.6</td>
<td></td>
</tr>
<tr>
<td>Bias</td>
<td>3.9 3.9 - 5.8 5.8 - -</td>
<td>8.6 - 8.6</td>
<td></td>
</tr>
</tbody>
</table>
| As can be, the improvement is remarkable compared to the earlier methods. It might be objected that it is a serious limitation of the least squares method that apriori values of the signal and noise degree variances have to be known. As the error degree variances for GOCE will be estimated quite accurately, the most crucial problem here is that the gravity


anomaly degree variances are not known. However, the least squares method is rather insensitive to the choice of weights. This is well known from other areas of geodesy. Let us illustrate this feature by means of the results from an error propagation, in which the uncorrelated error model with a very high standard deviation \( \sigma = 20 \text{ mGal} \) was used in the estimation of the least squares parameters, while the correlated model with the same standard deviation as before \( \sigma = 1 \text{ mGal} \) was utilised in the error propagation. The results are presented in the right part of Table 3. The insensitivity to the weighting is clearly illustrated. It can further be observed that the resulting estimator is insensitive to long-wavelength gravity anomaly errors at the same time as the truncation error is low. Thus, it seems suitable in the GOCE case to use the least squares method with pessimistic weights for the gravity anomalies.

8 CONCLUSIONS

When a GOCE satellite-only EGM, which is expected to be highly accurate up to a rather high maximum degree, will be combined with terrestrial gravity anomalies, it is important to apply a modification of Stokes’ formula that is insensitive to low-degree errors in the gravity anomalies. It is also important that the truncation error is small. In the error propagations presented in this paper, it was found that none of these requirements are fulfilled by the standard remove-compute-restore method, while the low-degree GOCE-only method fails because of its huge truncation error. The method most suitable for GOCE was found to be the (unbiased) least squares modification of Stokes’ formula. It was also demonstrated that the lack of a good apriori error model for the terrestrial gravity anomalies is not a big problem for a successful application of the method. In the GOCE case, a reasonably pessimistic error model of the terrestrial data seems like a good choice in matching various error sources in a least squares sense.

REFERENCES

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