GRAVITY FIELD DETERMINATION USING
MULTIRESOLUTION TECHNIQUES

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Abstract

In this paper we present results for modeling the Earth’s gravitational field using spherical wavelets and applying method-
omologies for the estimation of the corresponding coefficients. The observation types in our techniques could either be
gravity gradient tensor measurements from the Goce gradiometer, or other gravity mapping mission data such as the
Grace low-low intersatellite KA-band range-rate, or Champ high-low intersatellite GPS phase data, or a combination of
all the data types. Our approach allows both, either a wavelet-only solution or a combination of a spherical harmonics
part with an corresponding spherical wavelet part. Using appropriate techniques for the solution of the resulting normal
equation system, series coefficients up to a certain detail level can be estimated. Finally, we provide a demonstration of
the developed methodology using simulated data.

1 General Concept

Current gravity models of the Earth, such as EGM 96, are based on a series expansion in terms of spherical harmonics. In
order to create a new Earth gravity model this paper deals with an alternative representation of the geopotential, namely a
multi-resolution representation based on wavelet theory. Spherical harmonics are global basis functions and therefore not
appropriate for regional or local representations. Wavelet functions, however, are characterized by their ability to localize
both in the spatial and in the frequency domain. Hence, they are appropriate to model global, but also regional and even
local structures; for details see e.g. [1], [2].

Now we assume, that the real-valued signal (function) \( f(t) \) \( (r = \dot{r} = \text{geocentric position vector}, \ r = |t|, \ \dot{r} = \text{unit}

vector) is assumed to be harmonic, i.e. it fulfills the Laplacian differential equation \( \Delta f(t) = 0 \) outside a sphere \( \Omega_R \) with
radius \( R \). The solution of Dirichlet’s problem for the outer space of \( \Omega_R \) can be expressed by the spherical harmonics
representation

\[
f(t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} F_{n,m} Y_{n,m}(t),
\]

wherein \( Y_{n,m}(t) \) are the solid spherical harmonics of degree \( n \) and order \( m \). The Fourier coefficients \( F_{n,m} \) are charac-
terized by an optimal frequency (degree) localization, but they do not have any spatial information. Hence, we speak of
global parameters.

The computation of the Fourier coefficients requires preferable homogeneously distributed global data sets. Whereas
satellite data is almost globally distributed, terrestrial and airborne observations are always restricted to certain regions or
local areas. Consequently, we are interested in an alternative representation such as

\[
f(t) = \mathcal{F}(t) + \sum_{i=1}^{L} d_i h(t, t_i) + s(t).
\]

Herein \( \mathcal{F}(t) \) means a global trend model, e.g. a spherical harmonics expansion up to a low degree. In opposite to the global
parameters \( F_{n,m} \), the coefficients \( d_i \) are point parameters, since they are related to the \( L \) computation points \( P_i \equiv P(t_i) \).
To be more specific they represent the regional and local information extracted from the data by means of the “two-point”
function \( h(t, t_i) \). Furthermore, in Eq. (2) \( s(t) \) stands for the remaining unmodeled signal parts.

In order to compute the point parameters \( d_i \) of the alternative representation (2) we first replace the signal values \( f(t) \)
by the observations \( y(t_p) \), measured in the \( N \) observation points \( P(t_p) \), e.g. Goce measurements along the satellite
orbit. Next, we choose an appropriate kernel function such as the Shannon kernel or the reproducing kernel as well as an
appropriate system of points \( P \equiv P(t_i) \) with \( i = 1, \ldots, L \) for the computation of the parameters \( d_i \) collected in the \( L \times 1 \) vector \( \mathbf{d} \). This way we obtain the Gauss-Markoff model

\[
\mathbf{y} + \mathbf{e} = \mathbf{X} \mathbf{d} \quad \text{with} \quad \mathbf{D} = \sigma^2 \mathbf{P}^{-1},
\]

which is possibly not of full rank [3]. Herein \( \mathbf{y} \) means the observation vector, \( \mathbf{e} \) the vector of the corresponding measurement errors, \( \sigma^2 \) the variance of unit weight and \( \mathbf{P} \) the positive definite weight matrix. Additional operators, e.g. for the second radial derivative, have to be implemented into the coefficient matrix \( \mathbf{X} \).

Generally the least-squares solution of model (3) reads

\[
\tilde{\mathbf{d}} = (\mathbf{X}^T \mathbf{P} \mathbf{X})^{+} \mathbf{X}^T \mathbf{P} \mathbf{y} \quad \text{with} \quad \mathbf{D}(\tilde{\mathbf{d}}) = \sigma^2 (\mathbf{X}^T \mathbf{P} \mathbf{X})^{+},
\]

wherein \((\mathbf{X}^T \mathbf{P} \mathbf{X})^{+}\) means the pseudoinverse of the normal equation matrix. The estimator \( \tilde{\mathbf{d}} \) can be used to calculate both, estimators of the Fourier coefficients \( F_{n,m} \) up to a certain degree \( n = n' \) and an estimation of any spherical convolution \( (h \ast f)(t) \) with a band-limited kernel. Hence, an estimation of (2) may read

\[
\hat{f}(t) = \tilde{f}(t) + (h \ast \tilde{f})(t) = \sum_{n=0}^{n'} \sum_{m=-n}^{n} \tilde{F}_{n,m} Y_{n,m}(t) + \sum_{i=1}^{L} \tilde{d}_i h(t, t_i),
\]

wherein the kernel \( h(t, t_i) \) can be expanded as Legendre series

\[
h(t, t_i) = \sum_{n=n'+1}^{n''} \frac{2n + 1}{4\pi R^2} \left( \frac{R^2}{r^2_{r_i}} \right)^{n+1} H(n) P_n(r^T r_i)
\]

in terms of the Legendre polynomials \( P_n(r^T r_i) \) of degree \( n \). The Legendre coefficients \( H(n) \) reflect the frequency-behavior of the kernel.

## 2 Multiresolution Representation

The basic idea of the multiresolution representation (MRR) is to split a given input signal into a smoothed version and a certain number of detail signals by successive low-pass filtering. An MRR can be achieved in two steps, namely the decomposition of the signal into level-dependent coefficients (analysis) and the (re)construction by means of the detail signals (synthesis). Both the decomposition process and the (re)construction process can be illustrated by means of filter banks [4]. Filter banks can be implemented efficiently applying down- and upsampling strategies (pyramid schemes) [5].

In order to establish an MRR we identify the kernel \( h(t, t_i) \) in the Eqs. (2) and (6) with the scaling function \( \phi_{I+1}(t, t_i) \) of resolution level \( \text{scale} I + 1 \), namely

\[
\phi_{I+1}(t, t_i) = \sum_{n=n'+1}^{n''} \frac{2n + 1}{4\pi R^2} \left( \frac{R^2}{r^2_{r_i}} \right)^{n+1} \Phi_{I+1}(n) P_n(r^T r_i).
\]

Hence, the alternative representation (2) can be rewritten as an MRR, namely

\[
f(t) = \tilde{f}(t) + (\phi_{I+1} \ast \tilde{f})(t) + s(t) = \sum_{n=0}^{n'} \sum_{m=-n}^{n} F_{n,m} Y_{n,m}(t) + \sum_{i=1}^{I} g_i(t) + s(t).
\]

Herein on the right hand side the detail signals \( g_i(t) = (\tilde{\psi}_i \ast c_i)(t) \) are computed by the wavelet coefficients

\[
c_i(t_0) = (\tilde{\psi}_i \ast f)(t_0) = \int_{\Omega_{R}} f(t) \psi_i(t, t_0) d\omega(t) \simeq \sum_{l=1}^{L_i} d_{i,l} \psi_i(t_0, t_{i,l}),
\]

which are the components of the MRR. These scale-dependent point parameters are the counterpart to the Fourier coefficients of the spherical harmonics expansion (1). The Legendre coefficients \( \Psi_i(n) \) and \( \tilde{\Psi}_i(n) \) of the wavelet function of resolution level \( \text{scale} \ i \), i.e.

\[
\psi_i(t, t_i) = \sum_{n=n'+1}^{n''} \frac{2n + 1}{4\pi R^2} \left( \frac{R^2}{r^2_{r_i}} \right)^{n+1} \Psi_i(n) P_n(r^T r_i),
\]
and the dual function $\tilde{\phi}_i(t, t_1)$ are computed via the two-scale relation $\Psi_i(n) = \Phi_{i+1}(n) - \Phi_i(n)$ [6]. In general, a spherical harmonic wavelet function is always globally but mostly “quasi-compactly” supported. In our computations we use the Blackman function as scaling function $\phi_i(t, t_1)$ defined by the Legendre coefficients

$$\Phi_i(n) = \begin{cases} 
1 & \text{for } n = n' + 1, \ldots, 2n' - 1 \\
0.42 - 0.50 \cos \left( \frac{2\pi n}{2^{i+1}} \right) + 0.08 \cos \left( \frac{4\pi n}{2^{i+1}} \right) & \text{for } n = 2^{i-1}, \ldots, 2^i - 1 \end{cases}, \quad i \geq 0 \quad (11)$$

Note, that this function is derived from the definition equation of the Blackman window often used in signal analysis [5]. The Blackman wavelet is illustrated in Figure 1 for different scale values $i$. In the frequency domain these wavelets are compactly supported or strictly band-limited, respectively, i.e. only a few Legendre coefficients are not equal to zero (Figure 1c).

### 3 Estimation Concept

The computation of the $L_1 \times 1$ vector $\mathbf{d}_i = (d_{i,i})$ of scaling coefficients $d_{i,i}$ of highest resolution level $i$ from the observation vector $\mathbf{y}$ by parameter estimation methods according to the Gauss-Markoff model (3) means the initial step of the decomposition algorithm mentioned before. To be more specific we assume that the general solution reads

$$\hat{\mathbf{d}}_i = \mathbf{B}_i \mathbf{y} \quad \text{with} \quad D(\hat{\mathbf{d}}_i) = \mathbf{B}_i D(\mathbf{y}) \mathbf{B}_i^T, \quad (12)$$

wherein the matrix $\mathbf{B}_i$ may include the pseudoinverse of the normal equation matrix according to Eq. (4). In the following three steps we first compute the $L_i \times 1$ vector $\mathbf{d}_i = (d_{i,i})$ of level-1 scaling coefficients from the corresponding vector of level $i + 1$ by low-pass filtering

$$\tilde{\mathbf{d}}_i = \mathbf{H}_i \tilde{\mathbf{d}}_{i+1} = \mathbf{B}_i \mathbf{y} \quad \text{with} \quad D(\tilde{\mathbf{d}}_i) = \mathbf{B}_i D(\mathbf{y}) \mathbf{B}_i^T. \quad (13)$$

Next, the estimator $\mathbf{c}_i$ of level-$i$ wavelet coefficients $c_{i}(t_i)$ follows from band-pass filtering

$$\mathbf{c}_i = \mathbf{G}_i \tilde{\mathbf{d}}_i = \mathbf{A}_i \mathbf{y} \quad \text{with} \quad D(\mathbf{c}_i) = \mathbf{A}_i D(\mathbf{y}) \mathbf{A}_i^T. \quad (14)$$

Finally the estimator of the level-$i$ detail signal vector $\mathbf{g}_i$ is computed from

$$\mathbf{g}_i = \bar{\mathbf{G}}_i \mathbf{c}_i = \mathbf{E}_i \mathbf{y} \quad \text{with} \quad D(\mathbf{g}_i) = \mathbf{E}_i D(\mathbf{y}) \mathbf{E}_i^T. \quad (15)$$

The procedure providing the results of Eqs. (12) to (15) is characterized by the advantage that the computation of the estimators is completely embedded into

Figure 1: Blackman wavelet for different scale indices $i$

Figure 2: Level-3 Reuter grid with $L_3 = 278$ points. Reuter grids are non-hierarchical but equidistributed point systems on the sphere, i.e. the integration weights are independent on the position.
geodetic parameter estimation techniques. Thus, besides the estimators all the corresponding covariance matrices are computable. Additional statistical investigations like the estimation of confidence intervals and the testing of hypothesis for the parameters and for outliers can be performed easily.

4 Numerical Example

The estimation concept, presented in the last section, was applied to a simulated data set based on the EGM 96 gravity model. We computed disturbing potential values \( f(t) = T(t) \) up to degree \( n = 200 \) on a standard longitude-latitude grid at satellite altitudes randomly distributed between 450 km and 500 km. Furthermore, additional noise within \( \pm 0.5 \text{ m}^2/\text{s}^2 \) was considered. These values are collected in the observation vector \( \mathbf{y} \) of the Gauss-Markoff model (3). A diagonal weight matrix \( \mathbf{P} \) was chosen with purely latitude-dependent elements.

In order to compute an MRR of the given disturbing potential data we set \( I = 5 \) (see Figure 1c) and estimate \( \mathbf{d} := \mathbf{d}_5 = (\mathbf{d}_{5,i}) \) from the model (3) by means of the pseudoinverse according to (4). To be more specific, the coefficients \( \mathbf{d}_{5,i} \) with \( l = 1, \ldots, L_5 \) are related to a level-5 Reuter grid consisting of \( L_5 = 5180 \) points on the sphere \( \Omega_R \) with radius \( R = 6371 \) km. Figure 2 shows the level-3 Reuter grid with altogether \( L_3 = 278 \) points; see e.g. [1].

Figure 3 shows the five detail signals \( \mathbf{g}_i(t) \) according to Eq. (8) with \( i' = 1 \) on the sphere \( \Omega_R \). The sum of the corresponding five detail signals \( \mathbf{g}_i \) of levels \( i = 1, \ldots, 5 \) yields an approximation of the disturbing potential on the sphere \( \Omega_R \) (synthesis).

Basically a regional or local multiresolution representation can be estimated in the same manner. For this purpose both, the observation vector \( \mathbf{y} \) and the coefficient vector \( \mathbf{d} \) of the Gauss-Markoff model (3) are decomposed into two subvectors, namely \( \mathbf{y} = [\mathbf{y}_1 \ \mathbf{y}_2] \) and \( \mathbf{d} = [\mathbf{d}_1 \ \mathbf{d}_2] \). If e.g. the vector \( \mathbf{d}_1 \) stands for the coefficients related to the region of interest, the reduced normal equation system can be established. The neglect of the observations outside the region of interest, collected in the subvector \( \mathbf{y}_2 \), leads to edge effects. However, due to the localization property of the chosen scaling and wavelet functions these undesired effects are relatively small.

References


