A linear algorithm for computing the spherical harmonic coefficients of the gravitational potential from a constant density polyhedron

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Abstract. A new method is proposed to derive the spherical harmonic coefficients of the contribution to the gravitational potential of a constant density polyhedron of arbitrary shape. This method relies on a set of recurrent relationships between the involved integrals, and achieves a linear complexity in function of the number of edges of the polyhedron and the number of coefficients to be computed. In this paper we present the mathematical basis of the algorithm and discuss its possible applications.

Keywords: Gravitational potential, Polyhedron, Spherical harmonics

1 INTRODUCTION

The availability of gradiometric measurements along the tracks of the forthcoming low orbiting satellite GOCE will bring several new challenges to geodesists and geophysicists — such as the integration of this new kind of measurements in the gravity field determination techniques, the inversion of medium scale geophysical phenomena (such as seamounts, rifts, fracture zones) from this new gravity data, etc. In the context of the preparation of this satellite mission, we are interested in studying the sensitivity at various wavelengths (up to degree 400) of the gradiometric sensor to local geological structures, modeled as constant density polyhedra at a kilometric resolution.

In this paper we propose a new computational method for the determination of the spherical harmonic coefficients of the contribution to the potential of a constant density polyhedron of arbitrary shape. Up to now, this problem has been tackled by two different ways. On the first hand, some authors chose to first represent the radius of the surface of the considered body as an expansion into spherical harmonics, and then to introduce this expansion into the equations which define the coefficients of the potential. When the density is a simple function of the radius, the integration in the radial direction is possible. The remaining integrations were achieved either numerically [3] or analytically [1]. Such an approach is devoted to the computation of the low degrees: Balmino, for instance, gives the expressions of the coefficients up to degree 5 [1]. Moreover, the representation of the surface of the body through its harmonic development needs for this surface to be defined as a function of the radius, and thus the body to include the origin of the spherical coordinate frame: this voids the method in the case of local geological features.

On the other hand, one attempt to compute the coefficients through recurrent relationships was proposed by Werner [6]. His method relies on the well known stable recurrent relationships of the associated Legendre polynomials, applied to the computation of the integrals that define the coefficients. He derives new recurrent relationships between the coefficients of the monoms of the polynomial integrands, and uses the analytical values of each monom integral over each tetrahedron of the body. His method is designed to compute the expansion of the potential of local structures. However, due to the necessity to expand the polynomial integrands, the number of necessary operations behaves as the square of the number of coefficients to compute. The method is thus devoted to low degree expansions (his paper presents the coefficients of a sample computation up to degree 4).

Our method can be considered as affiliated to Werner’s approach. Just as him, we divide the polyhedral body into tetrahedra. Though, conversely to his work, we achieved to exhibit recurrent relationships between the values of the integrals that define the coefficients. These relations make our algorithm of linear complexity with regard to the number of coefficients to be computed (and of course to the number of tetrahedra), which allows to consider the computation of much higher degrees.

In the section 2, we present the mathematical basis of the proposed method. This algorithm has not yet been implemented. This latter task will be tackled in future works (section 3).
2 BASIS OF THE APPROACH

2.1 Notations

Spherical harmonics. The gravitational potential \( U \) of an extended body can be expressed as a series expansion in solid spherical harmonics of the form

\[
U (r, \theta, \lambda) = \frac{GM}{r} \left(1 + \sum_{n=1}^{+\infty} \left(\frac{a}{r}\right)^n \sum_{m=0}^{\infty} P_{n,m} (\cos \theta) [C_{n,m} \cos (m \lambda) + S_{n,m} \sin (m \lambda)]\right)
\]

(1)

where \( G \) is the gravitational constant, \( M \) is the total mass of the body, \( a \) is a reference distance, \((r, \theta, \lambda)\) are the spherical coordinates (radius, colatitude and longitude), \( P_{n,m} \) are the Legendre polynomials \((m = 0)\) and associated Legendre polynomials \((m \neq 0)\) as defined by Heiskanen and Moritz [2, p. 22-23].

The dimensionless coefficients \( C_{n,m} \) and \( S_{n,m} \) can be expressed as integrals in the form (after [2, p.59])

\[
\begin{bmatrix}
C_{n,m} \\
S_{n,m}
\end{bmatrix} = \frac{2 - \delta_{0,m} (n - m)!}{M (n + m)!} \int \int \int_{(r', \theta', \lambda') \in \text{body}} \left(\frac{r'}{a}\right)^n P_{n,m} (\cos \theta') \left[ \frac{\cos m \lambda'}{\sin m \lambda'} \right] \, dm
\]

(2)

where \( \delta_{0,m} \) is the Kronecker symbol, \( dm \) is the integration mass element, and where we use Werner’s matrix notation, as in [6].

Matrix \( h_{n,m} \). In this paper, we deal with the evaluation of the left member of equation 2 in the case when the considered body is a constant density polyhedron. In such a case, equation 2 becomes

\[
\begin{bmatrix}
C_{n,m} \\
S_{n,m}
\end{bmatrix} = k_{n,m} \int \int \int_{Q \in \mathcal{V}} h_{n,m} (Q) \, dv (Q)
\]

(3)

where \( k_{n,m} \) is a factor depending on the density of the body, its total mass, the degree \( n \) and the order \( m \), \( dv (Q) \) is the volume element at point \( Q = (r', \theta', \lambda') \), and where we define the \( 2 \times 1 \) matrix function \( h_{n,m} \) as

\[
h_{n,m} (Q) \equiv r^n P_{n,m} (\cos \theta') \left[ \frac{\cos m \lambda'}{\sin m \lambda'} \right]
\]

(4)

As far as the longitude is not concerned, \( h_{n,m} \) will play a similar part as a scalar function, keeping in mind that the written equations apply to the two components of the matrix. In such cases, we will also use the notation \( h_{n,m} \) to designate either of the two components of the matrix \( h_{n,m} \).

Coordinates. The spherical coordinates are convenient for some expressions, as in the case of the equation 4: in the following, we will drop the prime exponent which becomes useless, and note \( Q = (r, \theta, \lambda) \); we will also make use of the associated cartesian coordinates when necessary (see figure 1 (a)), and note them \( Q = (x, y, z) \)

\[
x = r \sin \theta \cos \lambda
\]

(5)

\[
y = r \sin \theta \sin \lambda
\]

(6)

\[
z = r \cos \theta
\]

(7)

We will note \((\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_\lambda)\) the unitary vectors of the local frame associated to the spherical coordinates, and \((\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z)\), the unitary vectors of the cartesian frame.

Tetrahedric element. Moreover, any polyhedron can be decomposed into a set of simplices (tetrahedra), each having one vertex at the origin and the opposite face taken from one of the polyhedral faces. The volume integral of equation 3 can then be expressed as the algebraic sum of the integrals over each simplex of the body \( \mathcal{V} \), where an integral is counted as positive if the outer normal to the polyhedron lays outside the simplex, and negative otherwise.

Hence, in the following, we will consider that the volume \( V \) is a simplex and note \((\sigma_k)_{k=0..3}\) its four faces, where \( \sigma_0 \) is the face that does belong to the polyhedron surface, and \((\sigma_k)_{k=1..3}\) are the faces which share the origin as common vertex. In concordance, the three edges of the face \( \sigma_0 \) will be noted \((\xi_k)_{k=1..3}\), where \( \xi_k \) is the common edge between face \( \sigma_0 \).
and face $\sigma_k$. We will also use the normal vector to each face $\sigma_k$, that will be noted $\mathbf{n}_k$, and the directing unitary vector (tangent vector) of each edge $\varepsilon_k$, that will be noted $\mathbf{t}_k$. Theses conventions are illustrated by the figure 1 (b).

In the following sections, we will show that the integral of the left member of equation 3, evaluated over a simplex, can be calculated through a set of recurrent relationships. We will note $H_{n,m}$ this integral

$$H_{n,m} = \iiint_{Q \in V} h_{n,m}(Q) \, dv(Q)$$

and use the notation $h_{n,m}$ for the integral of either component $h_{n,m}$ of the matrix $h_{n,m}$.

### 2.2 From volume to surface integral

Considering that

$$\nabla \cdot \left( \frac{1}{n + 3} r h_{n,m} \mathbf{u}_r \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{r^3 h_{n,m}}{n + 3} \right) = h_{n,m}$$

and applying the Green-Ostrogradsky theorem (divergence theorem), we get

$$H_{n,m} = \iiint_{V} h_{n,m} \, dv = \frac{1}{n + 3} \iint_{\sigma_0} r h_{n,m} \mathbf{u}_r . \mathbf{n}_0 \, d\sigma$$

The relation 10 gives an expression of the integral $H_{n,m}$ as the flow of a vector field through the sole face $\sigma_0$, using the fact that $\mathbf{u}_r$ is orthogonal to the other three normal vectors $(\mathbf{n}_k)_{k=1,3}$.

Besides, since face $\sigma_0$ is planar, we have

$$\forall Q \in \sigma_0, \; r(Q) \mathbf{u}_r(Q) . \mathbf{n}_0 = d_0$$

where $d_0$ is the distance between the origin and the plane of the face $\sigma_0$. Thus, we have

$$H_{n,m} = -\frac{d_0}{n + 3} \iint_{\sigma_0} h_{n,m} \, d\sigma$$

### 2.3 From surface to line integrals

We can then use the following identity

$$\nabla \wedge \begin{bmatrix} y h_{n,m} \\ -x h_{n,m} \\ 0 \end{bmatrix} = (n + m) \begin{bmatrix} x h_{n-1,m} \\ y h_{n-1,m} \\ z h_{n-1,m} \end{bmatrix} - (n + 2) \begin{bmatrix} 0 \\ 0 \\ h_{n,m} \end{bmatrix}$$

and apply the Stokes theorem. We get

$$\sum_{k=1}^{3} \int_{\varepsilon_k} \begin{bmatrix} y h_{n,m} \\ -x h_{n,m} \\ 0 \end{bmatrix} . \mathbf{t}_k \, dl = (n + m) \iint_{\sigma_0} r h_{n-1,m} \mathbf{u}_r . \mathbf{n}_0 \, d\sigma - (n + 2) \iint_{\sigma_0} h_{n,m} \mathbf{u}_z . \mathbf{n}_0 \, d\sigma$$
Using equations 10 and 12, equation 14 becomes

\[ H_{n,m} = \frac{d_0(n+m)}{(u_z \cdot n_0)(n+2)} H_{n-1,m} - \frac{d_0}{(u_z \cdot n_0)(n+2)(n+3)} \sum_{k=1}^{3} \int_{\varepsilon}^{t_e} \begin{bmatrix} yh_{n,m} \\ -xh_{n,m} \\ 0 \end{bmatrix} \cdot t_k dl \]  

(15)

The equation 15 shows that one can evaluate the integrals \( H_{n,m} \) through a recurrence on the degree \( n \), where each step consists in evaluating three line integrals along the edges of the face \( \sigma_0 \). In the following, we show that each line integral can itself be evaluated through a set of recurrence relations.

### 2.4 Evaluation of the line integrals

**Notations.** Let us consider the edge \( \varepsilon \) as representing any of the three edges of \( \sigma_0 \) and \( t = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \) its leading unitary vector. We define the curvilinear abscissa \( s \) along this edge by choosing as origin the point \( o = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix} \) of the straight line carrying \( \varepsilon \) which achieves the minimum distance to the origin of the spherical coordinate frame (illustrated by the point \( o_k \) on the figure 1). Let us call \( d \) this distance \((d^2 = o_x^2 + o_y^2 + o_z^2)\). Along \( \varepsilon \), we have

\[
\begin{align*}
  x &= o_x + t_x \cdot s \\
  y &= o_y + t_y \cdot s \\
  z &= o_z + t_z \cdot s \\
  r &= d^2 + s^2
\end{align*}
\]

(16)-(19)

The line integrals of equation 15 become

\[
\int_{\varepsilon}^{t_e} \begin{bmatrix} yh_{n,m} \\ -xh_{n,m} \\ 0 \end{bmatrix} \cdot t \, dl = (t_x o_y - t_y o_x) \int_{\varepsilon}^{t_e} h_{n,m} \, dl
\]

(20)

Let us note \( I_{n,m} \) the integral of the left member of equation 20, and \( J_{n,m} \), the associated \( 2 \times 1 \) matrix in concordance to previous notations. Let us also introduce two additional integrals in matrix form, \( J_{n,m} \) and \( K_{n,m} \), that will help in the forthcoming calculus:

\[
\begin{align*}
  I_{n,m} &= \int_{\varepsilon}^{t_e} h_{n,m} \, dl \\
  J_{n,m} &= \int_{\varepsilon}^{t_e} s h_{n,m} \, dl \\
  K_{n,m} &= \int_{\varepsilon}^{t_e} s^2 h_{n,m} \, dl
\end{align*}
\]

(21)

**Recurrence on \( I_{n,m} \).** The associated Legendre polynomials obey the following stable recurrence relation [4, p. 253]

\[
(n-m) P_{n,m}(\xi) = \xi (2n-1) P_{n-1,m}(\xi) - (n+m-1) P_{n-2,m}(\xi)
\]

(22)

By setting \( \xi = \cos \theta \) and multiplying equation 22 by \( r^n \begin{bmatrix} \cos m\lambda \\ \sin m\lambda \end{bmatrix} \) (where \( r \) and \( \lambda \) are functions of \( s \)), we get, after integration along \( \varepsilon \), and using relations 7, 18 and 19

\[
\begin{align*}
(n-m) I_{n,m} &= (2n-1) o_z I_{n-1,m} - (n+m-1) d^2 I_{n-2,m} \\
&+ (2n-1) t_z J_{n-1,m} - (n+m-1) K_{n-2,m}
\end{align*}
\]

(23)

**Recurrence on \( J_{n,m} \).** The calculus of the first derivative of \( h_{n,m} \) as function of \( s \) yields

\[
\begin{align*}
\left( d^2 + s^2 - (o_z + t_z s)^2 \right) \frac{d}{ds} h_{n,m} &= (n+m) \left( t_z d^2 - o_z s \right) h_{n-1,m} \\
&- n \left( t_z o_z - (1-t_z^2) s \right) h_{n,m} \\
&+ m \left( t_z o_y - t_y o_x \right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} h_{n,m}
\end{align*}
\]

(24)
By integrating both members of the equation 24 along $\varepsilon$, we get

$$ (n + 2) (1 - t_z^2) J_{n,m} = \left( (d^2 + s^2 - (a_z + z \cdot s)^2) h_{n,m} \right)_{s_{\text{max}}}^{s_{\text{min}}} + (n + 2) t_z o_z I_{n,m} - m \left( t_x o_y - t_y o_x \right) \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] I_{n,m} - (n + m) a_z d^2 J_{n-1,m} + (n + m) a_z J_{n-1,m} \quad (25) $$

where $s_{\text{min}}$ and $s_{\text{max}}$ are the bounds of the integration domain.

**Recurrence on $K_{n,m}$.** A similar method allows to get, from the expression of $\frac{d}{ds} (s h_{n,m})$, the following relation

$$ (n + 3) (1 - t_z^2) K_{n,m} = \left( (d^2 + s^2 - (a_z + z \cdot s)^2) s h_{n,m} \right)_{s_{\text{max}}}^{s_{\text{min}}} + (n + 4) t_z a_z J_{n,m} - m \left( t_x o_y - t_y o_x \right) \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] J_{n,m} - (n + m) t_z d^2 J_{n-1,m} + (n + m) a_z K_{n-1,m} \quad (26) $$

### 2.5 Initializations

Up to now, all the recurrent relationships use dependencies between the degrees $n$, with the order $m$ constant. Since $0 \leq m \leq n$, the recurrent computation has to be initialized with the values of $I_{n,m}$, $J_{n,m}$, $K_{n,m}$, and $H_{n,m}$ for every $n$ smaller than the maximal degree to be computed.

One can easily show that the relations 15, 25 and 26 do hold for $m = n$ by setting to null all the terms for which the order is strictly greater than the degree, and that the relation 23 does apply for $m = n - 1$ with the same convention.

As for the initialization of $I_{n,m}$, starting from the identity

$$ P_{n,n} (\xi) = (2n - 1) \sqrt{1 - \xi^2} P_{n-1,n-1} (\xi) \quad (27) $$

one can derive

$$ h_{n,n} = r^n P_{n,n} (\cos \theta) \left[ \frac{\cos m \lambda}{\sin m \lambda} \right] = (2n - 1) \sin \theta r^{n-1} P_{n-1} (\cos \theta) r \left[ \frac{\cos \lambda}{\sin \lambda} - \frac{\sin \lambda}{\cos \lambda} \right] \left[ \frac{\cos m \lambda}{\sin m \lambda} \right] = (2n - 1) \left[ \begin{array}{cc} x & -y \\ y & x \end{array} \right] h_{n-1,n-1} \quad (28) $$

and thus, with the equations 16 and 17, after intergration

$$ I_{n,n} = \left[ \begin{array}{cc} a_x & -a_y \\ a_y & a_x \end{array} \right] I_{n-1,n-1} + \left[ \begin{array}{cc} t_x & -t_y \\ t_y & t_x \end{array} \right] J_{n-1,n-1} \quad (29) $$

where $J_{n-1,n-1}$ depends only on $I_{n-1,n-1}$ through equation 25 (terms in $I_{n-2,n-1}$ and in $J_{n-2,n-1}$ being null).

### 2.6 Outline of the process

The previous sections showed step by step how we managed to build up the method. A detailed proof of all the used relations can be found in [5]. The figure 2 aims at giving a clearer view of the whole process. On the left are displayed the dependencies between the four integrals $I$, $J$, $K$ and $H$ during the initialization step, which computes the values of the integrals for $m = n$ (figure 2 (a)). Then, the integrals for $n > m$ can be computed using the dependencies shown on the right (figure 2 (b)). Each step of the recurrence (or each arrow on figure 2 — except the computation of $H$ which is simpler) only requires the evaluation of the function $h$ at the vertices of the tetrahedron: these values can also be obtained though recurrent relationships based on the equation 22.

The process can moreover by easily pipe-lined: the figure 2 (b) shows that part of the evaluation can be computed in parallel; or even fully parallelized throught the independent computation of the contributions of each tetrahedron of the considered body.
Figure 2: Recurrence dependencies according to the degree \( n \): (a) initialization recurrence for \( m = n \); (b) recurrence over degree \( n \) (order \( m \) constant).

3 FORTHCOMING WORK

In this paper, we have proposed a new algorithm to compute the coefficients of the spherical harmonic expansion of the potential of a constant density body, modeled as a polyhedron of arbitrary shape. This algorithm is based on a set of recurrent relationships that allows to achieve the computation in linear time with regard to the number to coefficients to be computed and the number of edges of the polyhedron, with a reasonable complexity factor (a few tens of operations at each recurrence step).

Designed to operate directly on the geometry of the polyhedron, and not on an expansion of its surface in spherical harmonics, it should fit the need for computing the coefficients of local geological structures, whose surface cannot be represented as a function of the distance to the origin. Moreover, its low complexity should allow to consider the computation of high degree expansions (up to degree 400) for rather complex bodies (thousands of faces), and thus to simulate the gravitational potential at wavelengths compatible — as far as the wavelength of the potential and the resolution of the source are concerned — with the satellite GOCE sensitivity.

Nevertheless, in the case of the Earth, the computation of the contribution of a local geological structure, located inside the crust, from the sum of the contributions of all its tetrahedra (with the center of the Earth as common vertex) might raise numerical issues. Besides, the stability of the recurrent process remains to be proven. Our further works will focus on these issues.

References


