MODELING THE EARTH'S GRAVITY FIELD FROM PRECISE SATELLITE ORBIT DATA: 
THE ACCELERATION APPROACH WORKS!

Pavel Ditmar, Alexis Van Eck van der Sluijs and Vladimir Kuznetsov

Department of Earth Observation and Space Systems (DEOS)
Delft University of Technology
Kluyverweg 1, 2629 HS Delft, The Netherlands
E-mail: ditmar@geo.tudelft.nl

ABSTRACT

Modeling the Earth’s gravity field from observed satellite accelerations is discussed. The emphasis is on deriving satellite accelerations from precise orbit data as well as on the optimal data weighting (including the case of time-dependent noise in orbit data). Numerical examples illustrate the importance of the proper data weighting. It is shown, in particular, that low-frequency noise in orbit data can be efficiently suppressed this way (contrary to the traditional approach based on the integration of variational equations).

1 INTRODUCTION

A number of dedicated satellite missions have been launched or will be launched soon in order to investigate the global gravity field of the Earth: CHAMP in 2000, GRACE in 2002, and GOCE in 2006 [14]. These missions are characterized by a different life-time and setup. But all the satellites involved are (to be) equipped with accelerometers to measure non-gravitational forces (e.g. the atmospheric drag) as well as with GPS receivers to determine the precise satellite orbit. Then, the collected information can be used to model the low-frequency part of the Earth’s gravity field. At present, two techniques for the computation of the gravity field from a satellite orbit are widely used: (i) the ”classical” technique based on the integration of the variational equations [13, 19, 10]; and (ii) the energy balance approach, which relates the kinetic energy of the satellite with the gravitational potential at a given point in compliance with the energy conservation law [9, 7, 17, 5, 4, 6]. Furthermore, a third technique - the acceleration approach - is currently under investigation [15, 3, 11, 12]. In this approach, the orbit-derived accelerations of a satellite are directly related to the Earth’s gravity field according to Newton’s second law. Ditmar and van Eck van der Sluijs [2] have already shown theoretically and with numerical examples that the acceleration approach is more accurate than the energy balance approach and is at least as accurate as the variational equations technique. In the course of last few months we have put efforts into the further development of this technique and its adaptation to real data processing. Some of the latest results are discussed in the current paper.

2 DATA PRE-PROCESSING: CONVERSION OF PRECISE ORBIT DATA 
INTO RESIDUAL SATELLITE ACCELERATIONS

The developed scheme for data pre-processing is presented in Fig. 1. More details about this scheme are given below.

2.1 Determination of total satellite accelerations

We find it essential that the satellite accelerations are derived from a kinematic orbit - the one determined directly from GPS measurements. Though a kinematic orbit may contain interruptions and relatively strong noise, it is, unlike a reduced-dynamic orbit, is not biased towards any existing gravity field model. Therefore, we fully control the stochastic properties of input data and can build the optimal procedure for their processing. As far as noise is concerned, it can be suppressed later in well-known ways (e.g. with a regularization).
Furthermore, we suggest that a simple 3-point scheme is applied to derive accelerations from an orbit:

$$\bar{a}(t) = \frac{x(t - \Delta t) - 2x(t) + x(t + \Delta t)}{(\Delta t)^2},$$

where $x(t)$ is a component of the satellite position vector at time $t$, $\Delta t$ is the data sampling interval, and $\bar{a}(t)$ is the derived acceleration. It is easy to show [2] that the accelerations $\bar{a}(t)$ in Eq. (1) are weighted averages:

$$\bar{a}(t) = \int_{-\Delta t}^{\Delta t} w(s) a(t + s) \, ds,$$

where the weight function $w(s)$ is a piece-wise linear function:

$$w(s) = \frac{\Delta t - |s|}{(\Delta t)^2}.$$  (3)

Our motivation to use only 3 orbit points for deriving accelerations is the following: (i) less information is lost in the presence of orbit interruptions and (ii) the covariance matrix of acceleration noise can be determined easier.
In order to keep the expressions simple, we assume that the accelerations are derived in the Celestial Reference Frame (CRF); otherwise, centrifugal and Coriolis terms have to be taken into account. Furthermore, we postulate that the functional model for computation of the Earth’s gravity field is defined in the local orbital reference frame (LORF). There two main reasons for that: (i) the consistency with non-gravitational acceleration measurements (the orientation of accelerometers on board is very close the LORF) and (ii) a dominant block-diagonal structure of the normal matrix.

Assuming that the kinematic orbit is given in the terrestrial reference frame (TRF), we can summarize that satellite accelerations are derived from the vector of orbit data $y$ according to the following expression:

$$d = R_{\text{CRF} \rightarrow \text{LORF}}D_{\text{TRF} \rightarrow \text{CRF}}y,$$

where $R_{\text{CRF} \rightarrow \text{LORF}}$ and $R_{\text{TRF} \rightarrow \text{CRF}}$ are rotation matrices (the names are self-explaining) and $D$ is the matrix of the double numerical differentiation. For each observational component, a fragment of matrix $D$ related to an (uninterrupted) series of observations is as follows:

$$D = \frac{1}{(\Delta t)^2} \begin{pmatrix} 1 & -2 & 1 \\ \vdots & \ddots & \vdots \\ 1 & -2 & 1 \end{pmatrix}$$

(5)

It is advisable to supply the derivation of average accelerations with a data screening procedure to remove possible outliers in the data.

2.2 Determination of reference satellite accelerations

In order to increase the quality of the obtained gravity field model, the reference accelerations due to all known sources have to be subtracted from the data. The remaining signal - residual accelerations - is inverted into spherical harmonic coefficients, which describe the correction of the global gravity field of the Earth w.r.t. to a reference model (e.g. EGM96). The final gravity field model is produced by summation of the obtained parameters and the parameters of the reference gravity field.

The reference accelerations are a sum of gravitational and non-gravitational accelerations.

2.2.1 Gravitational reference accelerations

Gravitational accelerations are determined by the static gravity field of the Earth’s as well as by time-varying phenomena. Point-wise accelerations related to the reference gravity field can be computed with well-known analytic expressions [see e.g. 2]. As far as the time-varying phenomena are concerned, only tide-related accelerations are taken into account so far. This can be done relatively easily thanks to well-developed theories of these phenomena [see e.g. 16]. It is likely that more effects will be taken into account later.

A necessary input for this computation is a set of precise satellite positions. We find it appropriate to use at this moment a reduce-dynamic satellite orbit. Such an orbit usually represents the satellite motion better than a reduced dynamic orbit. Furthermore, a reduced-dynamic orbit is usually uninterrupted or, at least, contains much fewer gaps than a kinematic one.

The set of computed gravitational point-wise accelerations has to be converted into a set of average accelerations. A tool to do this is a so-called averaging filter [2], which has to be applied in the celestial frame. Finally, the computed accelerations have to be converted into the LORF, i.e. the frame used for definition of the functional model.

2.2.2 Non-gravitational accelerations

The source of information about non-gravitational accelerations are accelerometers on board a satellite. The accelerometer readings are point-wise measurement. If their sampling rate coincides with or greater than that of the orbit data, the proper approach to handle non-gravitational accelerations is similar to that described in the previous section: (i) rotation into the CRF; (ii) application of the averaging filter, and (iii) rotation into the LORF. If the sampling rate is shorter than that of the orbit data, the averaging can also be done directly, i.e. by means of a numerical integration. In doing so, one should approximate the
integral of Eq. (2) with a sum:

$$\bar{a}(t) \approx \Delta s \sum_{i=1}^{n} w_i a_i,$$

where $n$ is the number of accelerometer data within the time interval $[t - \Delta t; t + \Delta t]$, $a_i$ are the accelerometer data, $w_i$ are the corresponding values of the weight function $w(s)$, and $\Delta s$ is the sampling rate of accelerometer data.

3 DATA PROCESSING: CONVERSION OF RESIDUAL SATELLITE ACCELERATIONS INTO AN EARTH’S GRAVITY FIELD MODEL

3.1 Building and solving the system of normal equations

The satellite accelerations (both point-wise and averaged) depend on the spherical harmonic coefficients linearly. Therefore, the following linear relationship can be written:

$$A\mathbf{x} = \Delta \mathbf{d},$$

where $\mathbf{x}$ is the vector of unknowns (geopotential coefficient corrections), $\Delta \mathbf{d}$ is the vector composed of residual average accelerations, and $A$ is the so-called design matrix. The optimal estimation of the vector $\mathbf{x}$ is given as the solution of the system of normal equations:

$$\hat{\mathbf{x}} = N^{-1} (A^T C_d^{-1} A)\mathbf{d},$$

where $N$ is the normal matrix:

$$N = A^T C_d^{-1} A + \alpha R$$

with the data noise covariance matrix $C_d$, the regularization parameter $\alpha$ and the regularization matrix $R$. Technically, there are numerous ways to solve the system of normal equations. In practice, we use two of them: (i) pre-conditioned conjugate gradient (PCCG) method and (ii) Cholesky de-composition; we refer for details to [2].

3.2 Data weighting

A crucial aspect of the proposed approach is a proper data weighting, which should be done on the basis of the covariance matrix $C_d$. Ditmar and van Eck van der Sluijs [2] have already presented a data weighting algorithm based on the assumption that noise in orbit data is white. A more general algorithm, which does not require such an assumption, is discussed below.

Data-weighting in the acceleration approach is a non-trivial operation because the covariance matrix is non-diagonal. Fortunately, it is not necessary to compute the inverse of the covariance matrix explicitly: it is enough to have an algorithm for application of $C_d^{-1}$ to vectors (namely, the search direction vectors in the PCCG method and the columns of the design matrix if the normal matrix is computed explicitly as required by the Cholesky de-composition).

According to Eq. (4), the covariance matrix $C_d$ can be explicitly represented as

$$C_d = R_{\text{CRF} \rightarrow \text{LORF}} C_d^{(\text{CRF})} R_{\text{LORF} \rightarrow \text{CRF}},$$

where $C_d^{(\text{CRF})}$ is the covariance matrix of noise in accelerations that are expressed in the CRF:

$$C_d^{(\text{CRF})} = D C_y^{(\text{CRF})} D^T,$$

with $C_y^{(\text{CRF})}$ being the covariance matrix of noise in positions that are expressed in the CRF:

$$C_y^{(\text{CRF})} = R_{\text{TRF} \rightarrow \text{CRF}} C_y^{(\text{TRF})} R_{\text{CRF} \rightarrow \text{TRF}}$$

Then, the expression for the inverse covariance matrix is:

$$C_d^{-1} = R_{\text{CRF} \rightarrow \text{LORF}} \left(C_d^{(\text{CRF})}\right)^{-1} R_{\text{LORF} \rightarrow \text{CRF}},$$

The subject of the further discussion is practical ways of applying the matrix $\left(C_d^{(\text{CRF})}\right)^{-1}$ to a vector or, in other words, data weighting in the CRF.
3.2.1 Approximate data weighting in the CRF

Computation of the matrix \( \left( C_d^{(CRF)} \right)^{-1} \) would be much simpler if the double differentiation matrix \( D \) had the inverse. Unfortunately, this is not the case. First of all, the number of rows in this matrix exceeds the number of columns [cf. Eq. (5)]. To cope with this problem, let us restore two missing rows (at the top and at the bottom) under the assumption that the matrix has to be circulant [1]:

\[
F = \frac{1}{(\Delta t)^2} \begin{pmatrix}
-2 & 1 & 1 & 1 \\
1 & -2 & 1 & 1 \\
& & \ddots & \ddots \\
1 & 1 & -2 & 1 \\
1 & 1 & 1 & -2
\end{pmatrix}
\]  

(14)

If the data set has gaps, this operation has to be repeated for each uninterrupted data fragment. The second problem is that the matrix we have obtained is singular: it has a zero eigenvalue that corresponds to the eigenvector \((1, 1, \ldots, 1)^T\). To solve this problem, let us (i) change the sign of all the elements in order to make the matrix semi-definite [as follows from Eq. (11), this operation does not change the matrix \( C_d^{(CRF)} \)] and (ii) regularize the matrix by adding a small number \( \epsilon^2 \) to its diagonal elements. Thus, we have a new matrix \( \tilde{F} \) with entries:

\[
\tilde{F} = \frac{1}{(\Delta t)^2} \begin{pmatrix}
2 + \epsilon^2 & -1 & -1 & -1 \\
-1 & 2 + \epsilon^2 & -1 & -1 \\
& & \ddots & \ddots \\
-1 & -1 & 2 + \epsilon^2 & -1 \\
-1 & -1 & -1 & 2 + \epsilon^2
\end{pmatrix}
\]  

(15)

To understand the second manipulation better, note that the application of a circulant matrix to a vector is nothing but a cyclic convolution, i.e. a filtering, which can be done either in the time domain or in the Fourier domain. In particular, application of the matrix \( D \) to a vector (i.e. the double differentiation) can be approximately represented in the Fourier domain as the multiplication of the vector Fourier transform with factor \(-\omega^2\). Then, the inverse operation is the division by \(-\omega^2\). Unfortunately, such a division is impossible for the zero frequency. A way out is to force the inverse filter to be limited at small frequencies. For instance, one may replace the factor \(-\omega^2\) with \(-\omega^2 - (\epsilon/\Delta t)^2\). It can be demonstrated [2] that such a replacement is fully equivalent to regularization of the matrix \( F \) as shown in Eq. (15). With such a regularization we postulate that noise in accelerations is not infinitesimal at any frequency, so that the weights assigned to different frequencies cannot be infinite.

For practical reasons, it is more convenient to do the inverse filtering in the time domain rather than in the Fourier domain. The explicit expression for the inverse filter, which approximates the application of the matrix \( E^{-1} \) to a vector, is as follows [2]:

\[
\left\{ \tilde{F}^{-1} \right\}_{ij} \approx \frac{\Delta t \tau}{2} e^{-\frac{|i - j| \Delta t}{\tau}},
\]  

(16)

where \( \tau \) is the filter half-width; it depends on the parameters \( \epsilon \) as \( \tau = \Delta t/\epsilon \).

Another operation to be considered is the application of the matrix \( \left( C_y^{(CRF)} \right)^{-1} \) to a vector. Fortunately, the matrix of noise in orbit data \( C_y^{(CRF)} \) has usually a simple structure. For example, noise in positions is frequently assumed to be non-correlated in time (this does not exclude correlations between 3 data components at each observation point). Then the matrix \( C_y^{(CRF)} \) is block-diagonal, with size of blocks being \( 3 \times 3 \). The inversion of such a matrix can be done with ease.

3.2.2 Exact data weighting in the CRF

Unfortunately, the approximate data weighting shown above may suffer from significant distortions at the edges of the data series. For this reason, we have also developed an exact procedure: the strict application of the matrix \( \left( C_d^{(CRF)} \right)^{-1} \) to a vector. The procedure is based on the PCCG method. Two basic operations are to be performed at each PCCG iteration: (i) the exact application of the matrix \( C_d^{(CRF)} \) to a vector and (ii) an approximate application of the matrix \( \left( C_d^{(CRF)} \right)^{-1} \) to a vector (pre-conditioning). In view of Eq. (11), the first operation is trivial. Furthermore, it can be easily adapted to a situation when
some data are removed from the data set (e.g. by the outlier detection procedure). In such a case, it is sufficient to eliminate from the matrix $C_a^{(CRF)}$ those columns and rows which correspond to removed observations (for more details, see [8, 2]). It is important to mention that a regularization of the covariance matrix remains essential in the exact data weighting scheme as well, so that non-zero elements of the matrix $D$ in a current row become equal to $(1, -(2 + \epsilon^2), 1)$. The influence of such a regularization onto the final results is discussed below. As far as the pre-conditioning is concerned, it can be performed with the approximate data weighting technique presented in the previous section.

4 NUMERICAL EXAMPLES

4.1 Comparison of the acceleration approach with the variational equations approach

In order to compare the proposed technique with a traditional one - variational equations approach - a numerical example has been considered. It is based on a satellite orbit that was computed by Dr. P. Visser (Aerospace Dept., Delft University of Technology) in compliance with the EGM96 gravity field model (Lemoine et.al, 1998) truncated at degree and order 50. The orbit parameters are as follows:

- Duration: 10 days
- Number of satellite revolutions: 161
- Mean inclination: $96.6^\circ$
- Radius: $6624 \pm 6$ km (the average elevation above the equator is $2466$ km)
- Sampling rate: 30s

Dr. P. Visser has also computed an auxiliary orbit on the basis of the JGM-3 gravity field model [18]. The discrepancies between the two orbits were numerically differentiated with the 3-point scheme, yielding a set of residual average accelerations.

Besides, we have also prepared three sets of noisy accelerations. To obtain them, we have contaminated the orbit discrepancies with noise. The following types of noise have been considered:

- White noise of 1-cm amplitude
- Low-frequency noise; it was generated as a sinusoidal function with 90-min period and 3-cm amplitude.
- The sum of white and low-frequency noise from the previous examples

Thus, we have obtained in total four sets of acceleration data, each of which was inverted into the gravity filed of the Earth. Three filter half-widths $\tau$ were considered in each case: 60 s, 180 s, and 600 s. Besides, the accelerations were inverted without any data weighting. For comparison, the data were also processed with the variational equations approach. To make the comparison of the techniques easier, no regularization was added to the normal matrix. Accuracy of all the produced models is presented in terms of rms and maximum geoid height errors (Table 1). One can see that all the techniques are very accurate if data contain no noise (the rms error does not exceed half-mm). In the presence of white noise, all the techniques also perform in a nearly the same way. An exception is the acceleration approach without a data weighting: the model accuracy in this case drops from 22-23 cm to 27 cm. The situation changes dramatically if the orbit data are contaminated with low-frequency noise. The acceleration approach without data weighting or with a short filter half-width is not sensitive to low-frequency noise: the rms errors remain at the sub-centimeter level. If, however, filter half-width increases, the model accuracy drops down. This is logical: a wider filter assigns larger weights to lower frequencies, where noise is present. The variational equations approach, which assumes that noise in orbit data is white (all the frequencies get a same weight), is the worst in this case: the rms error exceeds 20 cm. Finally, a combination of white and low-frequency noise is handled optimally by the acceleration approach with an average (180-s) filter half-width: the resulting rms error is 23 cm. The variational equations approach in this case leads to a 40-% increase of the rms error: up to 32 cm. The two latter computations are also compared in Fig. 2 as maps of geoid height errors.

4.2 Data weighting in the presence of time-dependent noise in orbit data

Noise in orbit data is frequently considered as non-correlated in time but time-dependent (non-stationary). The aim of the second numerical example is to demonstrate how important it is to apply a proper data weighting in the presence of non-stationary noise. The same satellite orbits have been used as in the previous example. A realization of non-stationary noise was
Table 1: RMS (max) geoid height errors after processing of noise-free and noisy orbit data with different approaches. The line with \( \tau = 0 \) corresponds to the acceleration approach without any data weighting.

<table>
<thead>
<tr>
<th>Data processing technique</th>
<th>Filter half-width ( \tau ) (s)</th>
<th>No noise</th>
<th>White noise 1-cm noise</th>
<th>Low-frequency noise 3-cm noise</th>
<th>Low-frequency noise + white noise</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acceleration approach</td>
<td>0</td>
<td>0.04 (0.3)</td>
<td>27.2 (123)</td>
<td>0.1 (0.4)</td>
<td>27.2 (123)</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>0.03 (0.2)</td>
<td>23.8 (103)</td>
<td>0.2 (0.8)</td>
<td>23.8 (104)</td>
</tr>
<tr>
<td></td>
<td>180</td>
<td>0.03 (0.2)</td>
<td>23.0 (97.1)</td>
<td>1.1 (2.9)</td>
<td>23.0 (98.0)</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>0.03 (0.2)</td>
<td>22.8 (98.4)</td>
<td>12.2 (32.2)</td>
<td>25.8 (119)</td>
</tr>
<tr>
<td>Variational equations approach</td>
<td>-</td>
<td>0.04 (0.2)</td>
<td>22.0 (108)</td>
<td>22.4 (66.4)</td>
<td>31.9 (122)</td>
</tr>
</tbody>
</table>

generated from a white noise realization of unit variance by multiplying each sample with a time-dependent variance \( \sigma_t^2 \). All the \( \sigma_t \) are in the interval between 0.5 cm and 5.0 cm with the average of 1.6 cm (these settings are based upon a real kinematic orbit of the CHAMP satellite, see Svehla & Rothacher, 2003).

The derived set of accelerations has been processed in three different ways:

1. Without any data weighting (matrix \( C_d \) is set equal to unit)
2. Without the time-dependent data weighting (matrix \( C_y^{CRF} \) is set equal to unit)
3. With the proper data weighting

No regularization was added to the normal matrix. The filter half-width \( \tau \) in the last two scenarios was set equal to 180 s. The obtained models are characterized by the rms geoid height errors of 40.5 cm, 36.1 cm, and 29.2 cm, respectively. Thus, a proper data weighting in the presence of non-stationary noise may increase the model quality up to 40%. The maps of geoid height errors for the first and last data processing scenarios are shown in Fig. 3.

5 CONCLUSION

Having considered some numerical examples, Ditmar and van Eck van der Sluijs [2] concluded that the acceleration approach is as good as the variational equations approach but faster. The latest studies have brought us to the conclusion that in a real data processing the acceleration approach can even be superior to traditional implementations of the variational equations approach. The acceleration approach may handle data with arbitrary distributed gaps. Furthermore, it can cope with a low-frequency noise in the data in a natural way: by a proper adjustment of the filter half-width in the data weighting algorithm. Besides, time-dependent accuracy of orbit data can easily be incorporated into the data processing procedure. Thus, the acceleration approach can improve the quality of Earth’s gravity models derived from satellite orbit data. This conclusion is also confirmed by first results of processing the real CHAMP satellite orbit. This is, however, the subject of a forth-coming publication.

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Fig. 2: Geoid height errors in the presence of white 1-cm noise and low-frequency 3-cm noise simultaneously: (a) Variational equations approach (rms error = 31.9 cm, max error = 122 cm); (b) Acceleration approach with \( \tau = 180s \) (rms error = 23.0 cm, max error = 98.0 cm).
Fig. 3: Geoid height errors when satellite accelerations are processed: (a) without any data weighing (rms error = 40.5 cm, max error = 183 cm); (b) with the proper data weighting (rms error = 29.2 cm, max error = 118 cm).
REFERENCES


