MULTISCALE MODELING FROM EIGEN-1S, EIGEN-2, EIGEN-GRACE01S, GGM01, UCPH2002_0.5, EGM96

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M. Gutting will present these results at the workshop.

Abstract

Spherical wavelets have been developed by the Geomathematics Group Kaiserslautern for several years and have been successfully applied to georelevant problems. Wavelets can be considered as consecutive band-pass filters and allow local approximations. The wavelet transform can also be applied to spherical harmonic models of the Earth's gravitational field like the most up-to-date EIGEN-1S, EIGEN-2, EIGEN-GRACE01S, GGM01, UCPH2002_0.5, and the well-known EGM96. Thereby, wavelet coefficients arise. In this paper it is the aim of the Geomathematics Group to make these data available to other interested groups. These wavelet coefficients allow not only the reconstruction of the wavelet approximations of the gravitational potential but also of the geoid, of the gravity anomalies and other important functionals of the gravitational field. Different types of wavelets are considered: bandlimited wavelets (here: Shannon and Cubic Polynomial (CuP)) as well as non-bandlimited ones (in our case: Abel-Poisson). For these types wavelet coefficients are computed and wavelet variances are given. The data format of the wavelet coefficients is also included.

Keywords: Multiscale Modeling, Wavelets, Wavelet Variances, Wavelet Coefficients, Gravitational Field Model Conversion

1 INTRODUCTION

During the last years spherical wavelets have been brought into existence (cf. e.g. [3], [4], [5] and the references therein). It is time to apply them to well-known models in order to offer easy access to the multiscale methods. During the last years spherical wavelets have been brought into existence (cf. e.g. [3], [4], [5] and the references therein). It is time to apply them to well-known models in order to offer easy access to the multiscale methods.

\[ \Phi_j(x, y) = \sum_{n=0}^{\infty} \varphi_j(n) \frac{2n + 1}{4\pi R^2} \left( \frac{R^2}{|x||y|} \right)^{n+1} P_n \left( \frac{x}{|x|}, \frac{y}{|y|} \right), \quad (x, y) \in \Omega_{ext}^R \times \Omega_{ext}^R, \]

where \( \Omega_{ext}^R \) denotes the outer space of the sphere \( \Omega_R \) of radius \( R \), \( P_n \) is the Legendre polynomial of degree \( n \) and \( \varphi_j(n) \) is called the symbol of the scaling function. This symbol is a sequence of numbers that determines the shape of the scaling function and has special properties (for details see [2], [3] or [4]). These properties make these kernels an approximation of the Dirac distribution that converges to it for \( j \) tending to infinity. Wavelets are defined by taking the difference of two consecutive scales which is performed by the refinement equation for the symbols:

\[ \psi_j(n) \tilde{\psi}_j(n) = \varphi_{j+1}^2(n) - \varphi_j^2(n), \quad n = 0, 2, 3, \ldots . \]

The primal and dual wavelets \( \Psi_j \), \( \tilde{\Psi}_j \) are obtained by either taking the square root or applying the third binomial formula to equation 2. Generally, they look the following way:

\[ \Psi_j(x, y) = \sum_{n=0}^{\infty} \psi_j(n) \frac{2n + 1}{4\pi R^2} \left( \frac{R^2}{|x||y|} \right)^{n+1} P_n \left( \frac{x}{|x|}, \frac{y}{|y|} \right), \]

\[ \tilde{\Psi}_j(x, y) = \sum_{n=0}^{\infty} \tilde{\psi}_j(n) \frac{2n + 1}{4\pi R^2} \left( \frac{R^2}{|x||y|} \right)^{n+1} P_n \left( \frac{x}{|x|}, \frac{y}{|y|} \right). \]
This construction allows an approximation of level $J$ of a potential $F$:

$$F_j = \Phi_j^{(2)} * F = \Phi_j^{(2)} * F + \sum_{j=j_0}^{J-1} \tilde{\Psi}_j * (\Psi_j * F) = \Phi_{j_0} * (\Phi_{j_0} * F) + \sum_{j=j_0}^{J-1} \tilde{\Psi}_j * (\Psi_j * F),$$

(5)

where "*" denotes the convolution in $L_2^2(\Omega_R)$. These convolution integrals can be discretized for numerical evaluation by methods presented in [1] or [4]. For the wavelet coefficients that we offer to other groups we chose the equiangular grid discussed in e.g. [1].

The important examples that we used for the computation of our multiscale representations are the Shannon type, the CuP type, and the Abel-Poisson type wavelets:

1.1 Shannon Wavelets

In the case of Shannon scaling functions the symbol $\varphi_j(n)$ reads as follows

$$\varphi_{j}^{SH}(n) = \begin{cases} 1 & \text{for } n \in [0, 2^j) \\ 0 & \text{for } n \in [2^j, \infty) \end{cases},$$

(6)

and for the corresponding wavelets we choose the P-scale version to resolve the refinement equation (2), i.e.

$$\tilde{\psi}_j^{SH}(n) = \psi_j^{SH}(n) = \sqrt{\left(\varphi_{j+1}^{SH}(n)\right)^2 - \left(\varphi_j^{SH}(n)\right)^2}.$$

(7)

1.2 Cubic Polynomial (CuP) Wavelets

In the CuP case the symbol takes the following form:

$$\varphi_{j}^{CP}(n) = \begin{cases} (1 - 2^{-j}n)(1 + 2^{-j+1}n) & \text{for } n \in [0, 2^j) \\ 0 & \text{for } n \in [2^j, \infty) \end{cases},$$

(8)

and for the corresponding wavelets we apply again the P-scale version.

1.3 Abel-Poisson Wavelets

For the Abel-Poisson scaling function the symbol takes the following form $\varphi_{j}^{AP}(n) = e^{-2^{-j}an}$, $n \in [0, \infty)$, with some constant $\alpha > 0$. We choose $\alpha = 1$. Since $\varphi_{j}^{AP}(n) \neq 0$ for all $n \in \mathbb{N}$ this symbol leads to a non-bandlimited kernel. It should be noted that the Abel-Poisson scaling function has a closed form representation which allows the omission of a series evaluation and truncation, and when constructing bilinear Abel-Poisson wavelets we want to keep such a representation as an elementary function. Thus, we decide to use M-scale wavelets whose symbols are deduced from the refinement equation (2) by the third binomial formula:

$$\psi_{j}^{AP}(n) = \left(\varphi_{j+1}^{AP}(n) - \varphi_{j}^{AP}(n)\right) \quad \text{and} \quad \tilde{\psi}_{j}^{AP}(n) = \left(\varphi_{j+1}^{AP}(n) + \varphi_{j}^{AP}(n)\right).$$

(9)

Since the Abel-Poisson scaling function and its corresponding wavelets are non-bandlimited we obtain just a good approximation by the numerical integration method based on an equiangular grid (we choose the parameter of polynomial exactness sufficiently large enough).

2 MULTISCALE REPRESENTATION OF THE GRAVITATIONAL POTENTIAL

The Earth’s gravitational potential $V$ in a point $x$ of the outer space of $\Omega_R$, i.e. the gravity potential $W$ without the part $\Phi$ caused by centrifugal force, possesses the following representation by convolutions with scaling functions and wavelets:

$$V(x) = \int_{\Omega_R} (\Phi_{j_0} * V)(y)\Phi_{j_0}(x, y)d\omega(y) + \sum_{j=j_0}^{J-1} \int_{\Omega_R} WT_j(V; y)\tilde{\Psi}_j(x, y)d\omega(y),$$

(10)
where $J_{max}$ is some suitably chosen maximal level of approximation and $WT_j(V; y) = (\Psi_j * V)(y)$ denotes the wavelet transform of $V$ at scale $j$ in the point $y$. In discrete form we get:

$$V(x) = \frac{GM}{R} \sum_{i=1}^{N_0} w_i^{j_0} v_i^{j_0} \Phi_{j_0}(x, y_i^{j_0}) + \frac{GM}{R} \sum_{j=j_0}^{J_{max}-1} \sum_{i=1}^{N_j} w_i^{j} \tilde{v}_i^{j} \tilde{\Psi}_j(x, y_i^{j}).$$

(11)

The weights of the integration are named $w_j$, $w_i$ and the corresponding knots are $y_j$, $y_i$. The scaling function coefficients $v_i^{j_0}$ and the wavelet coefficients $\tilde{v}_i^{j}$ result from the convolutions (12):

$$\frac{GM}{R} v_i^{j_0} = (\Phi_{j_0} * V)(y_i^{j_0}), \quad \frac{GM}{R} \tilde{v}_i^{j} = WT_j(V; y_i^{j}) = (\Psi_j * V)(y_i^{j}).$$

(12)

Fig. 1 and 2 show examplarily a part of the multiscale resolution of (10) where the details (Fig. 2) are added to the approximation of scale 7 (Fig. 1). By subtracting the non-centrifugal part of the ellipsoidal normal potential $V_{ell} = U - \Phi$ from $V$ the disturbing potential $T = V - V_{ell}$ can be obtained (see [7], [8] or [9]) and this subtraction can be performed for the coefficients $v_i^{j_0}$, $\tilde{v}_i^{j}$ in order to obtain a multiscale representation of the disturbing potential similar to (11), but with coefficients $t_i^{j_0}$ and $\tilde{t}_i^{j}$ that are related to $v_i^{j_0}$, $\tilde{v}_i^{j}$ by the equations (13):

$$\frac{GM}{R} t_i^{j_0} = \frac{GM}{R} v_i^{j_0} - (\Phi_{j_0} * V_{ell})(y_i^{j_0}), \quad \frac{GM}{R} \tilde{t}_i^{j} = \frac{GM}{R} \tilde{v}_i^{j} - (\Psi_j * V_{ell})(y_i^{j}).$$

(13)

A more detailed derivation can be found in [2].

3 FUNCTIONALS OF THE DISTURBING POTENTIAL

By virtue of the Bruns formula $N = T/\gamma$ (cf. [7], [8] or [9]) a multiscale representation of the geoid undulations can be computed from the multiscale decomposition of the disturbing potential, i.e. from the coefficients $t_i^{j_0}$ and $\tilde{t}_i^{j}$. Thereby, the normal gravity $\gamma$ is taken spherically as $\gamma = \frac{GM}{R^2}$. Thus, the geoid heights $N$ are described by

$$N(x) = R \sum_{i=1}^{N_0} w_i^{j_0} t_i^{j_0} \Phi_{j_0}(x, y_i^{j_0}) + R \sum_{j=j_0}^{J_{max}-1} \sum_{i=1}^{N_j} w_i^{j} \tilde{t}_i^{j} \tilde{\Psi}_j(x, y_i^{j}).$$

(14)

In Fig. 3 and 4 parts of a multiresolution of $N$ are presented (see [2] for a full multiresolution).
By definition the gravity disturbances correspond to the negative first radial derivative of $T$ which leads to the multiscale representation (15):

$$
\delta g(x) = -\frac{GM}{R} \left( \sum_{i=1}^{N_{m}} w_{i} \Phi_{j_{0}} \frac{\partial}{\partial r_{x}} \Phi_{j_{0}} (x, y_{i}^{j_{0}}) + \sum_{j=1}^{J_{\max} - 1} \sum_{i=1}^{N_{i}} w_{i} \frac{\partial}{\partial r_{x}} \tilde{\psi}_{j} (x, y_{i}^{j}) \right)
$$

$$
= \frac{GM}{R|x|} \sum_{i=1}^{N_{i}} w_{i} \Phi_{j_{0}} \Phi_{j_{0}} (x, y_{i}^{j_{0}}) + \frac{GM}{R|x|} \sum_{j=1}^{J_{\max} - 1} \sum_{i=1}^{N_{i}} w_{i} \tilde{\psi}_{j} \tilde{\psi}_{j} (x, y_{i}^{j})
$$

(15)

with

$$
\Phi_{j_{0}}^{\delta g}(x, y_{i}^{j_{0}}) = -|x| \frac{\partial}{\partial r_{x}} \Phi_{j_{0}} (x, y_{i}^{j_{0}}) = \sum_{n \neq 0} \varphi_{n}^{\delta g} (n) \frac{2n+1}{4\pi R^2} \left( \frac{R^2}{|x||y^{j_{0}}|} \right)^{n+1} P_{n} \left( \frac{x}{|x|}, \frac{y_{i}^{j_{0}}}{|y_{i}^{j_{0}}|} \right)
$$

(16)

where $\varphi_{n}^{\delta g}(n) = (n+1) \varphi_{j}(n)$ and the corresponding reconstructing wavelets $\tilde{\psi}_{j}^{\delta g}$ are constructed by applying the symbol $\tilde{\psi}_{j}^{\delta g}(n) = (n+1) \tilde{\psi}_{j}(n)$.

Analogously, the multiscale descriptions of the gravity anomalies $\Delta g = \delta g - \frac{2}{|x|} T$ and the vertical gravity gradients $g_{r} = \frac{\partial^{2} T}{\partial r^{2}}$ assume the following shape:

$$
\Delta g(x) = \frac{GM}{R|x|} \sum_{i=1}^{N_{i}} w_{i} \Phi_{j_{0}} \Phi_{j_{0}} (x, y_{i}^{j_{0}}) + \frac{GM}{R|x|} \sum_{j=1}^{J_{\max} - 1} \sum_{i=1}^{N_{i}} w_{i} \tilde{\psi}_{j} \tilde{\psi}_{j} \Delta g (x, y_{i}^{j}) ,
$$

(17)

$$
g_{r}(x) = \frac{GM}{R|x|^{2}} \sum_{i=1}^{N_{i}} w_{i} \Phi_{j_{0}} \Phi_{j_{0}} (x, y_{i}^{j_{0}}) + \frac{GM}{R|x|^{2}} \sum_{j=1}^{J_{\max} - 1} \sum_{i=1}^{N_{i}} w_{i} \tilde{\psi}_{j} \tilde{\psi}_{j} g_{r} (x, y_{i}^{j}) ,
$$

(18)

where

$$
\Phi_{j_{0}}^{\Delta g}(x, y_{i}^{j_{0}}) = \sum_{n \neq 0} \varphi_{n}^{\Delta g} (n) \frac{2n+1}{4\pi R^2} \left( \frac{R^2}{|x||y^{j_{0}}|} \right)^{n+1} P_{n} \left( \frac{x}{|x|}, \frac{y_{i}^{j_{0}}}{|y_{i}^{j_{0}}|} \right),
$$

(19)

$$
\Phi_{j_{0}}^{g_{r}}(x, y_{i}^{j_{0}}) = |x| \frac{\partial^{2}}{\partial r^{2}} \Phi_{j_{0}} (x, y_{i}^{j_{0}}) = \sum_{n \neq 0} \varphi_{n}^{g_{r}} (n) \frac{2n+1}{4\pi R^2} \left( \frac{R^2}{|x||y^{j_{0}}|} \right)^{n+1} P_{n} \left( \frac{x}{|x|}, \frac{y_{i}^{j_{0}}}{|y_{i}^{j_{0}}|} \right),
$$

(20)

with the symbols

$$
\varphi_{n}^{\Delta g}(n) = \varphi_{j}^{\delta g}(n) - 2 \varphi_{j}(n) = (n-1) \varphi_{j}(n) , \quad \varphi_{n}^{g_{r}}(n) = (n+1)(n+2) \varphi_{j}(n) ,
$$

(21)
and corresponding symbols $\tilde{\psi}^J_j(u)$ and $\tilde{\psi}^I_j(u)$ for the respective wavelets. As an example we show in Fig. 5 and 6 for the gravity anomalies $\Delta g$ again scale and detail 7 with CuP scaling functions or wavelets, respectively.

![Figure 5: $\Delta g$ at Scale 7 from EGM96 using CuP, [mgal].](image1)

![Figure 6: Wavelet detail at scale 7 of $\Delta g$ from EGM96 using CuP, [mgal].](image2)

### 4 WAVELET VARIANCES

The distribution of space-dependent signal energy of the disturbing potential is described by the scale and space variances of $T$ at scale $j$ and point $x$ (see [2], [5]):

$$\text{Var}_{j; x}(T) = \sigma^2_{j; x}(T) = \int_{\Omega_R} \int_{\Omega_R} T(y)T(z)\Psi_j(y, x)\Psi_j(z, x)d\omega(y)d\omega(z).$$

(22)

Using the wavelet coefficients $t^\Psi_j$ one can compute this quantity with the help of a Shannon kernel $SH(\cdot, \cdot)$ as

$$\sigma_{j; x}(T) = \frac{GM}{R}\sigma_{j; x}, \quad \sigma^2_{j; x} = \left( \sum_{i=1}^{N_j} w_i t^\Psi_j \, SH(y_i^j, x) \right)^2.$$ 

(23)

The wavelet variances of the geoidal heights $N$ are then given by $\sigma_{j; x}(N) = R\sigma_{j; x}$; and for the gravity disturbances, the gravity anomalies, and the vertical gravity gradients the variances can be obtained by a convolution with special Shannon kernels similar to (16), (19) and (20):

$$\sigma^2_{j; x}(\delta g) = \left( \frac{GM}{R|x|} \sum_{i=1}^{N_j} w_i^3 t^\Psi_j \, SH^\delta(y_i^j, x) \right)^2,$$

(24)

$$\sigma^2_{j; x}(\Delta g) = \left( \frac{GM}{R|x|} \sum_{i=1}^{N_j} w_i^3 t^\Psi_j \, SH^\Delta(y_i^j, x) \right)^2,$$

(25)

$$\sigma^2_{j; x}(g_r) = \left( \frac{GM}{R|x|^2} \sum_{i=1}^{N_j} w_i^3 t^\Psi_j \, SH^g(y_i^j, x) \right)^2.$$ 

(26)

Fig. 7 to 10 demonstrate the development of the energy distribution of the disturbing potential for $j = 4$ to 7.

### 5 WAVELET COEFFICIENTS

We supply to the end-user the scaling function or wavelet coefficients, $v_i^{\Phi;n}$ or $v_i^{\Psi;j}$ corresponding to the locations of the equiangular grid on $\Omega_R$ as well as the integration weights $w_i^{\Phi;n}$, $w_i^{\Psi;j}$. The wavelet coefficients are ordered as
a square matrix, in addition the weights form a first column. Comments are located in the first 20 lines of the file and are indicated by a % sign. The coefficients, a detailed model description and further figures can be found and downloaded at the following web page: http://www.mathematik.uni-kl.de/~wwwgeo/waveletmodels.html

References


