GRADIOMETRIC DATA ANALYSIS USING ICOSAHEDRAL GRIDS

Jürgen Kusche[(1)], Stephan Rudolph[(2)], Martin Feuchtinger[(2)], Karl Heinz Ilk[(2)]

[(1)] ***DEOS, Delft University of Technology***
Thijsseweg 11, 2629 JA Delft, The Netherlands
j.kusche@citg.tudelft.nl

[(2)] ***Institute for Theoretical Geodesy, University of Bonn***
Nußallee 17, 53115 Bonn, Germany
stephan@theor.geod.uni-bonn.de
martin@theor.geod.uni-bonn.de
ilk@theor.geod.uni-bonn.de

INTRODUCTION

The Gravity Field and Steady–State Ocean Circulation Explorer (GOCE) satellite mission will provide gravity field information of unprecedented resolution and precision. But the computation of GOCE gravity field models requires state–of–the–art numerical techniques, due to the vast amount of data and the number of unknowns. This holds even, if space–domain representations of the gravity field are implemented directly in the recovery process instead of the conventional spherical harmonic series. Unless very small regional gravity inversions are considered, the construction of preconditioners plays an important role. Another important issue are means to improve the stability of the least–squares problem. Space domain methods usually run into trouble for unnecessary fine discretizations.

Gradiometric data analysis with space–domain representation will play an important role for GOCE regional solutions, for calibration and data combination issues, or – if globally applied – for validation by a complete independent technique. A way to improve the stability of the resulting least–squares systems and to facilitate large–scale or global application is the use of icosahedral discretizations. This method as well as its combination with multigrid iteration techniques will be discussed in this contribution.

GRADIOMETRIC DATA ANALYSIS USING SPACE-DOMAIN REPRESENTATION

A space–domain representation of the anomalous gravity field is usually combined with a frequency–domain representation for modelling the known reference gravity field, i. e. a low–degree spherical harmonic expansion. The unknown gravity field can, for instance, be modelled by a surface density or gravity anomalies target function δg

$$V = \sum_{l=0}^{L_{\mathrm{ref}}} \sum_{m=-l}^{l} V_{lm}^{\mathrm{ref}}\; Y_{lm} \;+\; \frac{R}{4\pi} \int_{\Omega} S(\cdot, Q)\; \delta g(Q)\; d\omega \tag{1}$$

with spherical harmonics Y_{lm}, the mean radius R of the earth, and extended Stokes–Function S. In this context, "anomalies" are defined with respect to the chosen reference field and to a reference surface (Bjerhammar sphere). The observables of GOCE gradiometry, gravity gradients $\mathbf{\Gamma}_{ij}$ measured in an satellite–fixed frame, can thus be related to the unknown anomalies by

$$\mathbf{\Gamma} = \boldsymbol{U}\boldsymbol{V}\boldsymbol{U}^T \qquad\qquad \boldsymbol{V}_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} V^{\mathrm{ref}} \;+\; \frac{R}{4\pi} \int_{\Omega} \frac{\partial^2}{\partial x_i \partial x_j} S(\cdot, Q)\; \delta g(Q)\; d\omega\;. \tag{2}$$

Here $\boldsymbol{U}$ is the rotation matrix between earth–fixed and satellite–fixed frame. Finally, for the numerical treatment and least–squares formulation of the problem, the target function δg in (1) has to be discretized using some kind of finite element base functions. The method as described above has been investigated in various studies, see e. g. [2] and the references therein, for different mission concepts and different observation types including gradiometry as well as GPS. If the unknown target function represents gravity anomalies, the approach is known as "inversion of the Stokes Integral". In [4] this method has been used to study the influence of the polar gaps and combination with airborne/terrestrial data for GOCE gravity anomaly recovery.

Space–domain representations like (1) come with advantages but also with drawbacks. From the idea of the method it is clear that it can be applied both regionally and globally. Regional application means there is less numerical burden with normal system accumulation, de–correlation of observations, and solution of the system, since we have to deal with fewer unknowns. Consequently no further simplifications on the orbit or data sampling need to be imposed. The method, although strictly speaking a “space–wise” approach, can account for coloured noise models when the data is given as a time–series as shown in [4]. For global applications, however, the same limitations than for other space–wise techniques hold. Combination with terrestrial data is generally simple, and heterogeneous resolution can be achieved without problems. On the other hand, precautions must be taken against wavelength errors and geographical truncation errors. The ill–conditionedness of the normal system due to polar data gaps and downward continuation, as well as means to overcome this problem, have been discussed in [4] and [6].

When regarding a high–resolution mission like GOCE, some challenging numerical tasks have to be faced. Two problems, which deserve special attention in our gravity recovery method, will be covered in the next two sections: 1. The normal equation systems are especially bad conditioned if one aims at the determination of gravity anomalies defined on a conventional equi–angular geographical grid. 2. The normal equation systems are dense and efficient preconditioners for their efficient solution have to be designed.

ICOSAHEDRAL PARTITIONING OF THE SPHERE

If the target function in (1) is globally discretized by means of a conventional equi–angular spherical grid, one runs into problems due to the strong concentration of meshes near the north and south pole. An unnecessary fine discretization causes strong ill–conditionedness, especially since the data will be sparse in these regions. In [4] this problem has been circumvent by a 90°-rotation of the coordinate system. This, however, can only improve the stability of regional solutions and does not help for global gravity field recovery. A more flexible way out of this problem would be the use of (nearly) equi–area triangulations of the reference sphere Ω, i. e. the determination of gravity anomalies defined on non–conventional grids. Among others, icosahedral grids can be used.

Icosahedral grids are derived easily from an initial icosaheder (Fig. 1, left) by consecutive sub–division of the sides connecting neighbouring edges (see fig. 2). The edges of an icosaheder (or 12–edge polyhedron) constitute an initial level–0 grid on a prescribing sphere, with the sides of the 20 triangular equi–sized meshes being great circle segments. Fig. 1 (right) shows the global level–4 grid. These derived level–n grids do not strictly share the equi–area property with the original level–0 grid. But the deviations are small, as can be seen from table 1. Table 1 shows the number of unknowns for a global discretization on certain levels, as well as the minimum and maximum size of a triangular mesh. For GOCE, with an expected spherical harmonic resolution between 250 and 300, the levels 5 and 6 seem to be appropriate.

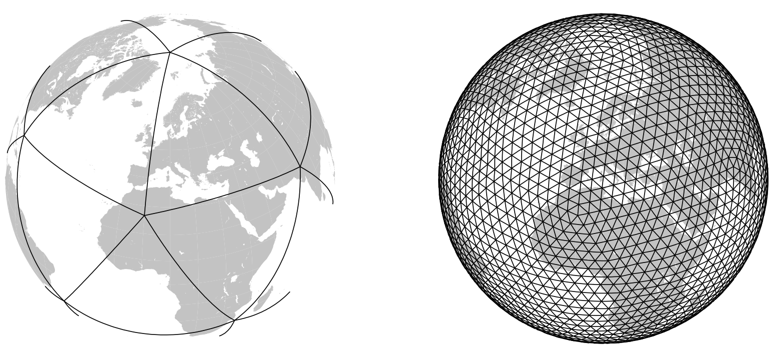

Figure 1: Level–0 grid (icosaheder, left) and level–4 grid (right)

Icosahedral grids for the representation of the anomalous gravity field have been used e. g. in studies regarding the gravimetric boundary value problem. Combination of icosahedral grids with multigrid iteration techniques was first proposed by the authors of [7] for the numerical solution of the shallow–water equations on the sphere. As in geodesy, the conventional long–established technique in this field is to model the governing equations by spherical harmonics.

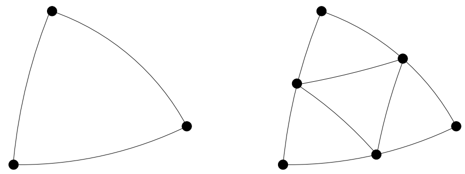

Figure 2: Consecutive subdivision

		$(\Delta\omega)^{1/2}$ [deg]	
Level	N_T	min	max
4	5120	2,61	3,12
5	20480	1,30	1,56
6	81920	0,65	0,78

Table 1: Level statistics. N_T is the number of triangles, $(\Delta\omega)^{1/2}$ would be the side lenght $\Delta\lambda = \Delta\phi$ of an equi–angular equatorial block of the same area.

MULTIGRID ITERATION TECHNIQUE

After choosing a certain grid level for the discretization, setting up observation equations, de–correlation or weighting of observations, and accumulation of the normal matrix one ends up with the normal equation system

$$\boldsymbol{N}^{\alpha}\boldsymbol{x} = \boldsymbol{y} \ , \tag{3}$$

where $\boldsymbol{N} = \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} + \alpha\boldsymbol{K}$ denotes the (α–regularized) normal matrix, $\boldsymbol{y} = \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{l}$ the right–hand side vector, and $\boldsymbol{x}$ the vector of unknown gravity anomalies. $\boldsymbol{K}$ is a suitable regularization matrix. Methods for the choice of α and $\boldsymbol{K}$ are discussed in [4] and [6]. The problem now is that in global or large–scale regional gravity recovery from GOCE the normal system will be large–sized and dense. Moreover, optimized regularization techniques require the system to be solved for a multitude of possible values of α, thus for various condition numbers. An iterative solution reads

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \boldsymbol{C}^{\alpha}(\boldsymbol{y} - \boldsymbol{N}^{\alpha}\boldsymbol{x}^k) \ , \tag{4}$$

where the preconditioning matrix $\boldsymbol{C}^{\alpha}$ should approximate $(\boldsymbol{N}^{\alpha})^{-1}$. Usually it is more efficient to implement the preconditioner in the conjugate gradient method instead of making direct use of the linear iteration (4). Unfortunately, for the task of GOCE gravity field recovery with (1) at high resolution, most preconditioners fail to work efficiently. A way out of this problem is the use of multigrid iteration techniques. For a general background on multigrid see [1]. Let the solution we want to obtain from gradiometric data be discretized on a level–n grid. The basic idea is that an approximate solution of the residual system in (4) can be computed cheaply by applying standard relaxation methods. Since relaxation methods yield approximations with smooth errors, a correction can be efficiently calculated on the coarser level–$(n-1)$ grid. This step indeed involves the solution of the restricted normal system for the unknowns defined on the coarser grid. But the idea can be used recursively by employing the even coarser level–$(n-2)$ grid. Only on the coarsest grid, say level–n_0, a direct solution has to be computed for the system

$$\boldsymbol{R}\boldsymbol{N}^{\alpha}\boldsymbol{R}^T\boldsymbol{x}_{n_0} = \boldsymbol{R}\boldsymbol{y} \ . \tag{5}$$

Here $\boldsymbol{R}$ is a sparse rectangular matrix which connects the base functions between the level–n grid and the level–n_0 grid. Note that every time we “step down” one level of the icosahedral grids, the number of unknowns decreases by a factor of 4. The idea is illustrated in fig. 3. This process, usually called multigrid iteration, serves therefore as a fast solver of the level–n equations and simultaneously solves the level–$(n-1)$ and level–$(n-2)$

equations down to the level n_0. It can also be used as a preconditioner in the conjugate gradient method. Thus, in principle a whole sequence of gravity field solutions of different resolution can be obtained in the same process. In practice, it turns out that for badly conditioned – even if regularized – problems the best performance will be obtained with employing not more than 3 grids. More details and an algorithmic description are given in [3].

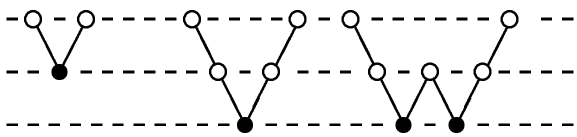

Figure 3: Possible cycles in multigrid iteration. A cycle is an iteration step consisting of relaxation or smoothing (∘) and coarse–grid correction (•)

There exists a vast amount of mathematical literature on multigrid methods, see [1]. For gravity field recovery using space–domain representations these techniques have been shown to work efficiently by [3], [5], and [6].

SIMULATION EXAMPLE

In order to verify the concepts mentioned above and their applicability to the GOCE mission, a number of simulations are currently carried out. In this paper we restrict ourselves to a simple example, to explain the general set–up. An area with strong gravity signal and strong spatial variations was considered. See the fig. 4 for the "true" gravity anomalies from the EGM96 model. Then, for this area (plus an additional "overlap" of approximately 8° in each direction to prevent for truncation errors), the representation (1) was discretized using a portion of the level–5 grid. This means the unknowns of the problem were 2992 mean gravity anomalies each representing a triangular area, which would correspond in size to an equitorial block of approximately 1.4° × 1.4°. The result is shown in fig. 5.

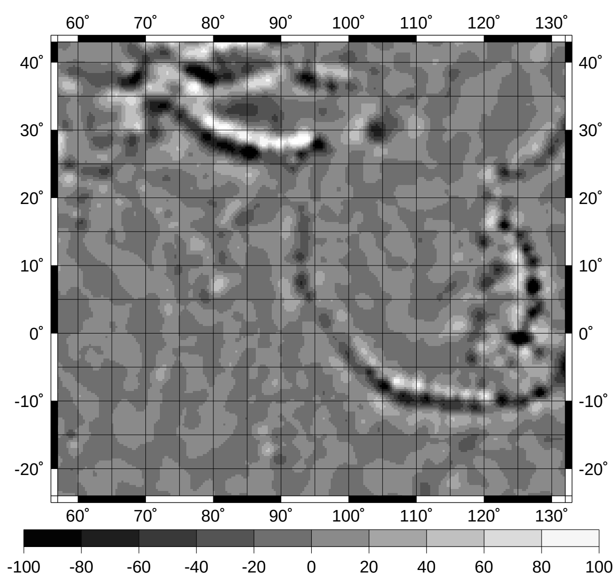

Figure 4: "True" gravity anomalies [mGal] with respect to the OSU91a reference field ($L_{\text{ref}} = 36$)

The simulated data set was provided by K. H. Ilk, J. Kusche, and P. Visser in the framework of the IAG–Special Commission 7 ("Satellite Gravity Field Missions") and is general accessible via the internet home–page `http://www.geod.uni-bonn.de/SC7`. It consists of a 30 days "pure gravitational", nearly spherical ($e = 0.001$, $h = 250km$), non–polar GOCE–orbit, which was used without further simplification for the setup of the observation equations. Along this orbit gravity gradient data was simulated for every $5sec$. All three diagonal

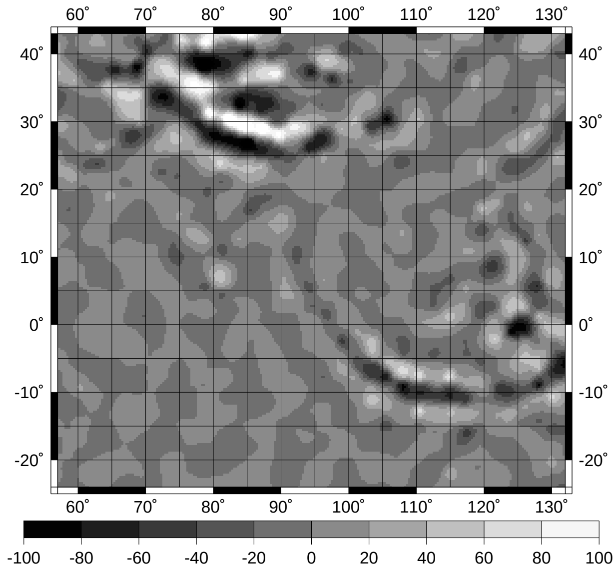

Figure 5: Recovered gravity anomalies [mGal] with respect to the OSU91a reference field ($L_{\text{ref}} = 36$)

components (along–track, cross–track, radial) have contributed in this example. All three components have been artificially contaminated by coloured noise. As the "true" gravity field model for data simulation we used the EGM96 coefficient set complete up to degree and order $L = 300$.

CONCLUSIONS

The use of non–conventional, icosahedral grids facilitates a stable formulation of the least–squares problem of space–domain gravity recovery from gradiometric data. Moreover, it can be combined with fast multigrid iteration techniques for the construction of preconditioners. This will enable space–domain gravity recovery for GOCE on large–scale regional or even global scale.

REFERENCES

[1] W. Hackbusch, *Multi–Grid Methods and Applications*, Springer, Berlin, 1985

[2] K. H. Ilk, R. Rummel, and M. Thalhammer, "Refined Method for the Regional Recovery from GPS/SST and SGG", in *Study of the Gravity Field Determination using Gradiometry and GPS (Phase 2)* CIGAR III/2 Final Report, ESA contract No. 10713/93/F/FL, 1995

[3] J. Kusche, "Implementation of multigrid solvers for satellite gravity anomaly recovery" in press

[4] J. Kusche and K. H. Ilk, "The Polar Gap Problem", in *From Eötvös to Milligal*, Final Report ESA Study, ESA/ESTEC Contract No. 13392/98/NL/GD, 2000

[5] J. Kusche and S. Rudolph, "The Multigrid Method for Satellite Gravity Field Recovery", in *Geodesy beyond year 2000*, Springer, Berlin, 2000

[6] S. Rudolph, *Regionale und globale Gravitationsfeldanalyse hochauflösender Satellitendaten mittels Mehrgitterverfahren*, Deutsche Geodätische Kommission, Reihe C, Nr. 528, Munich, 2000

[7] G. R. Stuhne and W. R. Peltier, "New icosahedral grid–point discretizations of the shallow water equations on the sphere", *Journal of Computational Physics*, vol. 148, pp 23–58, 1999