TIME-WISE DETERMINATION OF THE EARTH'S GRAVITY FIELD FROM SGG OBSERVATIONS

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INTRODUCTION

The objective of our research is to develop algorithms and software for the efficient inversion of satellite gravity gradiometry (SGG) data, which are to be collected during the GOCE mission. The Earth's gravity field to be determined this way is represented as a series of spherical harmonics. There exist two different approaches to the inversion: the so-called *space-wise* approach, when computations are performed in an Earth-fixed reference frame, and the *time-wise* approach, when all the computations are carried out in the orbit-oriented frame [5]. Our focus is on the time-wise approach. A first attempt of the time-wise inversion was done in our group more than one year ago. Since then, we have significantly improved the inversion algorithms by adopting a more accurate functional model and increasing the efficiency of the implementation.

According to the time-wise approach, the measured gravity gradients are related to the potential coefficients describing the Earth's gravity field in the following manner:

$$V_{zz}(r,\omega_o,\omega_e,I) = \frac{GM}{R^3}\sum_{l=0}^{L_{max}}(l+1)(l+2)\left(\frac{R}{r}\right)^{l+3}\sum_{m=0}^{l}\left\{C_{lm}\sum_{k=-l[2]}^{l}F_{lmk}(I)\times \left[\begin{array}{c}\cos\Phi_{km}\\ \sin\Phi_{km}\end{array}\right]_{l-m:\ \mathrm{odd}}^{l-m:\ \mathrm{even}} + S_{lm}\sum_{k=-l[2]}^{l}F_{lmk}(I)\left[\begin{array}{c}\sin\Phi_{km}\\ -\cos\Phi_{km}\end{array}\right]_{l-m:\ \mathrm{odd}}^{l-m:\ \mathrm{even}}\right\}, \quad (1)$$

where r, ω_o, ω_e, and I are parameters describing the satellite orbit and position (see Fig.1); GM is the gravitational constant of the Earth; R is the equatorial radius of the Earth; $\Phi_{km} = k\omega_o + m\omega_e$; $F_{lmk}(I)$ are the inclination functions [3, 7]; L_{max} is the maximum degree of the model; C_{lm} and S_{lm} are the unknown potential coefficients to be found during the inversion;

$$\left[\begin{array}{c}A\\ B\end{array}\right]_{l-m:\ \mathrm{odd}}^{l-m:\ \mathrm{even}} = \begin{cases} A & \text{if } l \text{ and } m \text{ have the same parity}\\ B & \text{if } l \text{ and } m \text{ have different parity}\end{cases}$$

Note that equation (1) describes the functional model for the V_{zz} component of the gravity tensor. Similar equations can be derived for the other relevant components to be measured during the mission, namely V_{xx}, V_{yy}, and V_{xz} [5].

Equation (1) can be written in matrix form as

$$\mathbf{d} = \mathbf{A}\tilde{\mathbf{x}}, \quad (2)$$

where $\tilde{\mathbf{x}}$ is the model vector (i.e. the vector of unknown coefficients); $\mathbf{d}$ is the data vector; and $\mathbf{A}$ is the design matrix. As such, the least-squares estimate of $\tilde{x}$ is given by

$$\mathbf{x} = (\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}\mathbf{d}, \quad (3)$$

where $\mathbf{P}$ is the so-called weight matrix; it is introduced in order to take into account colored noise in the data. Strictly speaking, (3) should also contain a regularization term, but we have omitted it, because the regularization is beyond the scope of the article.

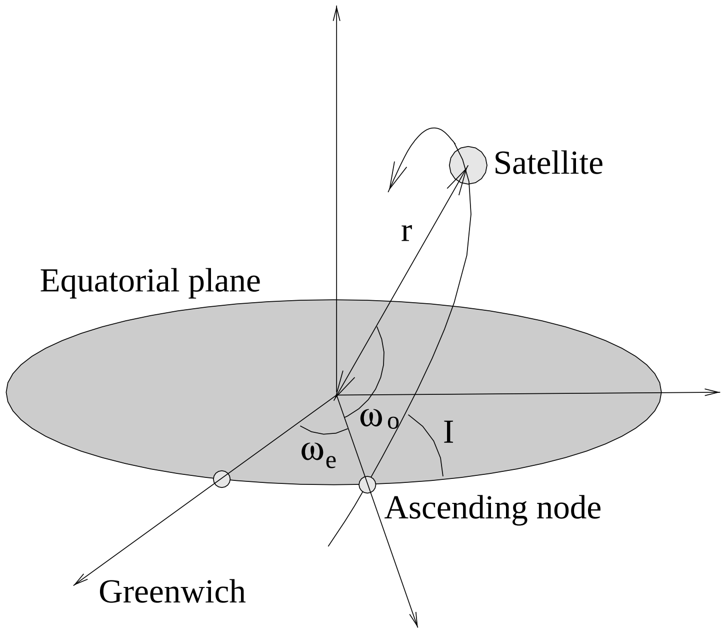

Fig. 1: The satellite orbit and position are determined by 4 parameters: I - orbit inclination; ω_e - angular distance between the Greenwich meridian and the ascending node of the orbit; ω_o - angular distance between the ascending node and the satellite itself; r - distance between the Earth's center and the satellite; for a non-Keplerian orbit, the instant value of a parameter should be taken.

According to (3), the solution can be found by solving the system of linear normal equations. There are many ways to do so; our preference is the conjugate gradient method [2]. In the simplest form, the method can be written as follows:

1. $\mathbf{x_0} = \mathbf{0}$

2. $\mathbf{q_0} = \mathbf{r_0} = \mathbf{A}^T\mathbf{Pd}$

3. $k = 0$

4. $\mathbf{a_k} = \mathbf{A}^T\mathbf{PA}\,\mathbf{q_k}$

5. $\alpha_k = \dfrac{\mathbf{r_k}^T\mathbf{q_k}}{\mathbf{a_k}^T\mathbf{q_k}}$

6. $\mathbf{x_{k+1}} = \mathbf{x_k} + \alpha_k\mathbf{q_k}$

7. $\mathbf{r_{k+1}} = \mathbf{r_k} - \alpha_k\mathbf{a_k}$

8. If $\|\mathbf{r_{k+1}}\| < \epsilon$ stop

9. $\beta_{k+1} = \dfrac{\mathbf{r_{k+1}}^T\mathbf{r_{k+1}}}{\mathbf{r_k}^T\mathbf{r_k}}$

10. $\mathbf{q_{k+1}} = \mathbf{r_{k+1}} + \beta_{k+1}\mathbf{q_k}$

11. $k = k + 1$, go to item (4),

where $\mathbf{x_k}$ is a model vector; $\mathbf{r_k}$ is a residual vector; $\mathbf{q_k}$ is a model update vector; $\mathbf{a_k}$ is an auxiliary vector; α_k and β_k are auxiliary scalars; ϵ is a small value; index "k" denotes the current iteration step.

From this scheme it is obvious that the most time-consuming step in the course of inversion is step (4). It contains two most tedious operations: multiplication of matrix $\mathbf{A}$ with a vector and multiplication of matrix $\mathbf{A}^T$ with a vector. Unfortunately, the total size of matrix $\mathbf{A}$ is so huge, that it cannot be stored in the computer memory. The only solution is to calculate the matrix "on-the-fly", simultaneously with the multiplication. It means that the entire matrix $\mathbf{A}$ should be computed twice at each iteration, which makes the inversion even more time-consuming. Thus, in order to solve efficiently the inverse problem, one must be able to apply quickly and accurately matrices $\mathbf{A}$ and $\mathbf{A}^T$ to a vector. In order to speed up the computations, one can introduce various approximations. Naturally, the approximation errors have to be controlled, i.e. they must not distort any noticeably the estimated potential coefficients. Note that the application of $\mathbf{A}$ to the vector $\mathbf{q_k}$ is nothing else but a synthesis step, i.e. solving the direct problem for a given model vector $\mathbf{q_k}$. That is why, in the following we discuss how to optimize the synthesis step, which is defined by (1) and (2). All the ideas related to the synthesis are also applicable to the multiplication of matrix $\mathbf{A}^T$ to a vector.

It is worth while to mention that in practice we use a more complicated modification of the conjugate gradient method than written above. Firstly, we incorporated pre-conditioning [1], which tremendously accelerates the convergence. Secondly, we included Schönauer's smoothing [6].

OPTIMIZATION OF THE SYNTHESIS STEP

The main goal of the optimization of the synthesis step is to find the optimal balance between accuracy and speed of computations. That is we tried to make computations more accurate when that could noticeably improve the quality of the inversion. At the same time, we introduced more approximations if they could improve the performance without distorting the inversion results.

Improvement of the Inversion Accuracy

Various numerical examples demonstrated that the time-wise approach, as it was implemented originally, resulted in rather inaccurate potential-coefficient estimates. We revealed a number of reasons for that. One of the dominant reasons was related to the assumption that the orbit inclination I is constant. The assumption was made because computation of the inclination functions $F_{lmk}(I)$ for every observation point, as suggested by (1), would be too time-consuming. As a solution, we propose to apply the Taylor's expansion to the inclination functions:

$$F_{lmk}(I) = F_{lmk}(I_0) + F'_{lmk}(I)\Big|_{I=I_0}(I - I_0) + \frac{1}{2}F''_{lmk}(I)\Big|_{I=I_0}(I - I_0)^2 + \dots, \tag{4}$$

where I_0 is the mean orbit inclination. Numerical examples showed that in fact it is sufficient to retain only the first two terms in the expansion (4). Thus, instead of computing the inclination functions at every point, one can simply calculate the inclination functions and their derivatives at the mean inclination.

Improvement of the Inversion Efficiency

The inversion software in its original form was very time consuming, and models with maximum degree and order $L_{max} \approx 180$ or more could not be solved for. A number of improvements have been undertaken both at the mathematical and at the software level. One of the most important innovations proposed was to apply the Taylor's expansion to the radius-dependent factor in (1).

The synthesis step, if implemented in the straightforward way, contains the following sequence of nested loops:

```
Loop over data {
  Loop over "l" {
    Loop over "m" {
      Loop over "k" {

        ...

      }
    }
  }
}
```

The total number of operations in this case can be estimated as $O(NL^3_{max})$, where N is the number of observations. Let's apply now Taylor's series expansion:

$$\left(\frac{R}{r}\right)^{l+3} = \left(\frac{R}{r_0}\right)^{l+3} \left[1 + (-l-3)\frac{r-r_0}{r_0} + \frac{1}{2}(-l-3)(-l-4)\left(\frac{r-r_0}{r_0}\right)^2 + \ldots\right], \tag{5}$$

where r_0 is the mean orbit radius. Each term in this expansion may be represented as the product of two factors: one depending on l and the other depending on r (i.e. on the measurement point). Therefore, if any particular term in the series is considered, the l-dependent factor can be brought outside the loop over data. The same holds for other l-dependent factors in (1). As a result, the loops over "l" and over data may be de-coupled. Thus, if the number of terms in the series (5) is constant, the total number of operations can be estimated as $O(NL^2_{max})$. In practice, it is sufficient to retain 6 to 8 terms. Numerical studies showed that the inversion performance improves in this case rather significantly.

We compared the current inversion software with that developed earlier from the viewpoint of their performance. In case of a model with L_{max}=80, the new implementation is a factor of 10 faster than the former one. This is in spite of an increased number of operations due to taking the time-varying inclination into account.

Time-wise Semi-analytical Inversion

In [4] the so-called iterative semi-analytical (TWSA) approach to inversion of SGG data was analyzed. The TWSA iterations converge to the following solution:

$$\mathbf{x} = (\mathbf{A_0}^T\mathbf{PA})^{-1}\mathbf{A_0}^T\mathbf{Pd}, \tag{6}$$

where $\mathbf{A_0}$ is an approximation of matrix $\mathbf{A}$. Expression (6) is very similar to (3) with the exception that matrix $\mathbf{A^T}$ is replaced with matrix $\mathbf{A_0^T}$, hence the conjugate gradient method still can be used to find the solution. The replacement does not introduce any bias, i.e. the "true" solution could be obtained this way if the data were noise-free. In the presence of noise, however, the solution changes: it becomes non-optimal, i.e. deviates from that defined by the least-squares method. A natural way to define matrix $\mathbf{A_0}$ is to retain only zero-order terms in series (4) and (5). In other words, both the orbit inclination and the orbit radius are assumed to be constant. On one hand, this accelerates the matrix-vector multiplications at least 10 times. On the other hand, matrix $\mathbf{A_0}$ remains sufficiently similar to matrix $\mathbf{A}$.

NUMERICAL EXAMPLES

Two numerical examples are considered, both being based on the same data set. 3 components of the SGG tensor (V_{xx}, V_{yy}, and V_{zz}) are simulated on the basis of a realistic GOCE orbit (6614 km $< R_{orbit} <$ 6632 km). The chosen sampling rate is 5 seconds; the mission duration is 29 days. The model used for the simulation is OSU91a complete up to degree and order 180. The data have been contaminated with colored noise. The noise applied to components V_{xx} and V_{zz} is 3 mE/$\sqrt{(\text{Hz})}$ above the frequency 27 cycles per revolution and increases as 1/f at lower frequencies. The noise added to the V_{yy} component is two times less. The inversion is performed on the CRAY T3E super-computer with 64 processing elements.

In the first example, the simulated data set is inverted using the time-wise approach. The difference between the calculated and the "true" gravity field is expressed in terms of geoid height misfits and shown in Fig. 2. The rms misfit is about 1.5 cm; the maximum misfit does not exceed 6 cm (the polar areas beyond latitudes ±70° are not considered). The wall-clock time of computations is about 7 hours. As expected, most of the time (2.5 hours and 3.8 hours) is spent for the application of matrices $\mathbf{A}$ and $\mathbf{A^T}$, respectively, to vectors.

In the second example, the inversion is done by means of the TWSA approach. The differences between the estimated potential coefficients and the "true" ones are again expressed in terms of geoid height misfits and shown in Fig. 3. One can see that these differences are more than 10 times smaller than the accuracy of the models obtained (the maximum difference does not exceed 0.5 cm, whereas the rms difference is less than 0.1 cm).

The total wall-clock time of the inversion in the second example is only 2.8 hours. The higher efficiency compared with the time-wise approach is due to the use of matrix $\mathbf{A_0^T}$ instead of matrix $\mathbf{A^T}$ in multiplications. This step

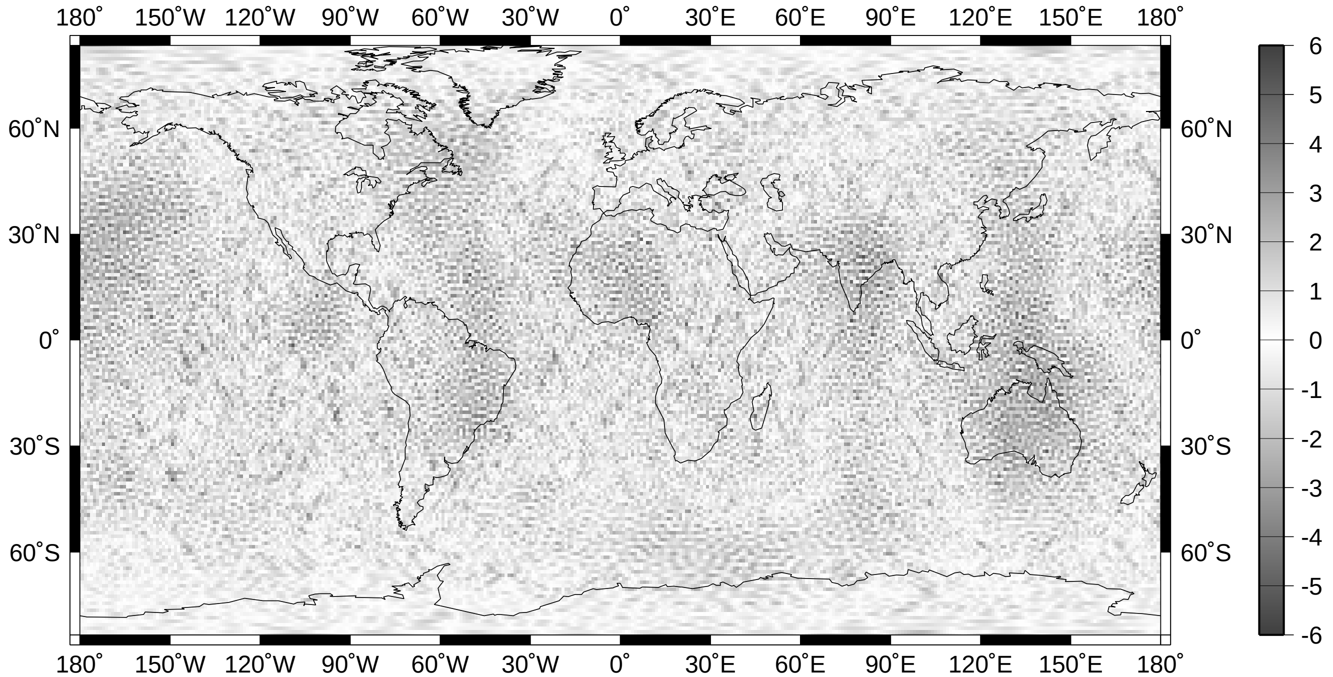

Fig. 2: Differences between "true" and estimated potential coefficients in terms of geoid height errors (cm) for the time-wise approach.

takes now only 20 minutes. Multiplications of matrix **A** with vectors take about the same time as in the "full-scale" time-wise approach.

It is worthwhile to mention that we expect further improvement of the inversion accuracy due to a more elaborated choice of the regularization.

CONCLUSIONS

A new software for the time-wise inversion of SGG data has been developed. The performance of the implementation in terms of accuracy and efficiency has been improved significantly. The new software has been applied so far to modeling the Earth's gravity field up to degree and order 180. However, there are no doubts that models up to the maximum degree and order 300 can be computed in this way. An additional acceleration of the inversion process can be achieved if the time-wise semi-analytical approach is followed.

The rms accuracy of the models obtained so far (on the basis of realistic mission scenarios) is about 1.5 cm in terms of geoid heights. The time-wise and the time-wise semi-analytical approach give about the same results. Whether the TWSA approach is a real alternative to the time-wise approach for maximum degree and order more than 180 has still to be investigated.

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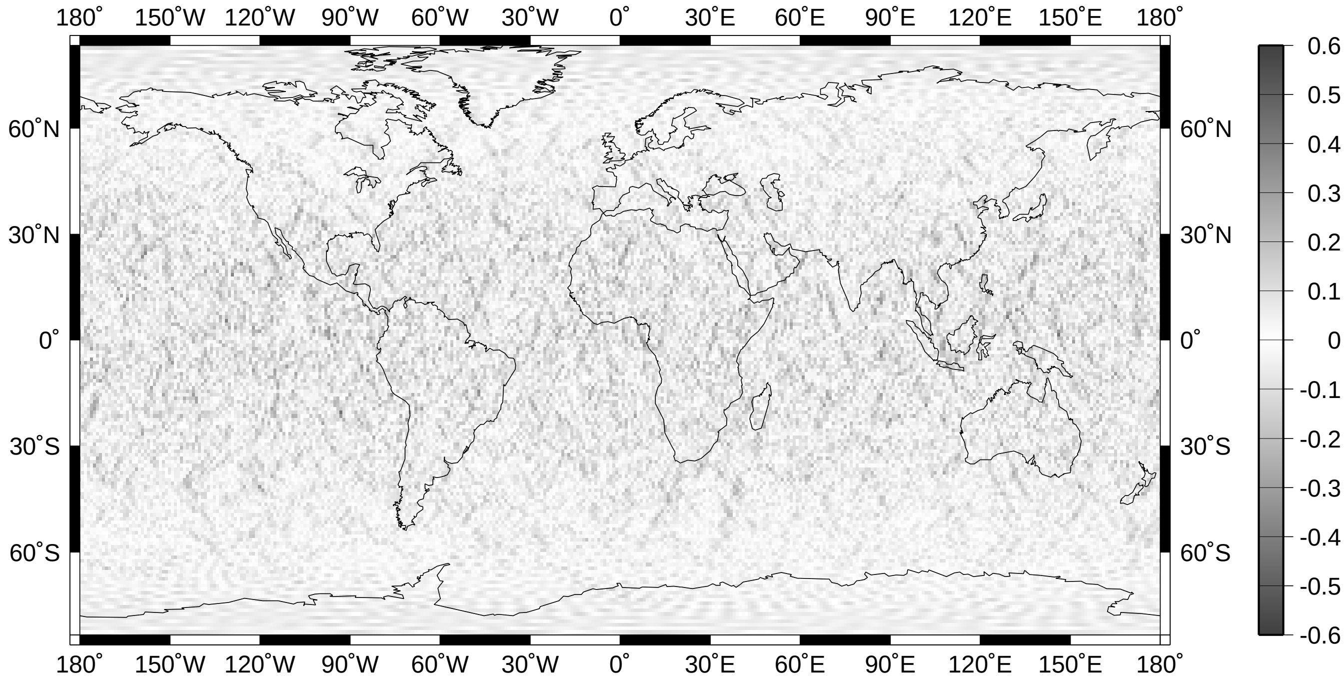

Fig. 3: Geoid height differences (cm) between the time-wise and the time-wise semi-analytical solution.

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