HONEY WE SHRUNK THE INTERFEROGRAM MATRIX!

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Covariance matrix

- The Covariance Matrix estimation is a key issue in the framework of coherent SAR analysis of scenarios characterized by the presence of distributed targets.
- It is possible to use it to better estimate target elevation and displacement information.

We want to estimate it as best as we can to improve the reliability of our estimates.
Sample Estimator

The standard statistical method is to gather statistically homogenous samples and compute the sample covariance matrix.

Advantages:
- Easy to compute
- Being unbiased

Disadvantages:
- Contains a lot of statistical errors
- Ill conditioned when the number of samples is comparable with the number of images
Structured Estimator

Another possibility is to use a completely structured estimator.

A structured estimator is a parametric estimator where we have to define a model and estimate its parameters.

**Advantages:**
- Reduce statistical errors
- Full rank matrix

**Disadvantages:**
- Can bias the estimates
- Tends to be misspecified
- Can be computationally intensive
In this work we suggest the shrinkage estimator

\[ \hat{W} = \delta M + (1 - \delta)S \]

- The aim of the *shrinkage* estimator is to collect only the advantages of the previous two estimators.
Optimal Shrinkage Intensity

- The optimal shrinkage intensity is the one minimizing

\[ \hat{\delta} = \arg \min_{\delta} \left\{ \| \delta \mathbf{M} + (1 - \delta) \mathbf{S} - \mathbf{W} \|^2 \right\} \]

- It depends on the true (unknown) covariance matrix.

- It’s possible to solve this difficulty by finding a consistent estimator, asymptotically equivalent to the optimal one.

\[ \hat{\delta} = \frac{\sum \sum \text{var}(s_{ij}) - \text{cov}(m_{ij}, s_{ij})}{\sum \sum \text{var}(m_{ij} - s_{ij}) + \left| E[m_{ij}] - s_{ij} \right|^2} \]
Statistical moment evaluation

- We only have a data-set of $N_L$ observations, assumed to be i.i.d. and coming from an unknown population
  $Y = [y_1, y_2, \ldots, y_{N_L}]$
- We use a bootstraping method: it estimates the sampling distribution of an estimator by constructing a number of resamples of the observed data-set, by random sampling with replacement
  $Y = [y_1, y_2, y_3, \ldots, y_{N_L}]$
  $Y_1 = [y_1, y_1, y_2, y_4, \ldots, y_{N_L}]$
  $Y_2 = [y_2, y_3, y_4, y_4, \ldots, y_{N_L}]$
  $Y_{N_{\text{boot}}} = [y_1, y_2, y_2, y_5, \ldots, y_{N_L}]$
- So, from the original data-set we generate $N_{\text{boot}}$ data-set, each one with its sample and structured covariance matrix, suitable to numerically estimate the necessary statistical moments
SKP Structured model

Consider the case where the same scene is imaged $M$-times with $N$ different baselines, then we have $NI = NM$ total images.

Let $y(b, t)$ denote the pixel acquired with normal baseline $b$ and temporal baseline $t$, the expected value of each interferogram is:

$$E[y_1(b_1, t_1) y_2^*(b_2, t_2)] = \sigma^2 \gamma_1(b_1, t_1, b_2, t_2) = \sigma^2 \gamma_s(b_1, b_2) \gamma_t(t_1, t_2)$$

**hypothesis of separability**

Back-scattered power  
Spatial decorrelation  
Temporal decorrelation
SKP Structured model (2)

- From the previous hypothesis it follows that the covariance matrix is given by:

\[ \mathbf{W} = \mathbf{E}[\mathbf{y}\mathbf{y}^H] = \mathbf{T} \otimes \mathbf{B} \]

- And in case of multiple independent targets:

\[ \mathbf{W} = \mathbf{E}[\mathbf{y}\mathbf{y}^H] = \sum_k \mathbf{T}_k \otimes \mathbf{B}_k \]

The advantage of the Sum of Kronecker Product (SKP) Decomposition is due to the existence of a fast technique for the decomposition of any matrix into a weighted SKP.
Numerical Results
Correct Model

We first construct a covariance matrix as one KP (one matrix is \([5 \times 5]\) and the other one is \([3 \times 3]\)), and use the correct model to construct the structured estimator.

When we use the correct model the structured estimator is always the best estimator.
We generate a covariance matrix as the sum of 2 KP, but, also in this case, we construct the structured covariance matrix using only one KP.

If the model is misspecified the shrinkage improves the estimate.
DInSAR framework

- As the satellite does not acquire data from all the possible geometries (normal baselines) every time, we get a different structure for the coherence matrix.

\[ \Gamma = \gamma_{SNR} \cdot \Gamma_t \circ \Gamma_s \]

- Important to point out this structure as the Hadamard Product (HP) is a sub-matrix of the KP.
DInSAR Metodology

In this case we introduce another hypothesis, that is also a fundamental constraint:

Spatial and temporal decorrelation must change slowly with the change of the baselines

1. We can interpolate the coherence in correspondence of all the possible combinations among temporal and spatial baseline reconstructing the KP
2. Then we apply the SKP Decomposition and truncate the result at the first KP assuming a single target
3. We generate the HP using the first term of the SKP Decomposition to construct the structured coherence matrix
Covariance matrix from HP

Spatial decorrelation matrix

Temporal decorrelation matrix

Resulting coherence matrix
DInSAR Metodology

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From HP to KP

Thanks to interpolation we reconstruct the Kronecker Product structure

$NI \times NI$
DInSAR Metodology

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SKP Decomposition

The first term of the SKP Decomposition

Estimated spatial decorrelation matrix

Estimated temporal decorrelation matrix
DInSAR Metodology

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We generate data statistically represented by the coherence matrix previously shown. For different number of looks we compute the optimal shrinkage intensity using the asymptotic formulation and use the bootstrapping to estimate the necessary statistical moments.
Optimal Shrinkage Intensity
Frobenius norm of the estimation errors
Conclusions

- In this work we proved how we should not trust blindly in the observed data.
- This is the reason we suggest a shrinkage estimator when a-priori information is available.
- In particular, we focused on matrices based on SKP model, the advantages are:
  1. Fast SKP Decomposition
  2. Possibility to get a structured model using only the hypothesis of separability
  3. Suitable for matrices constructed as the HP of two matrices with a low-pass behavior
Thank you