A GENERAL FORMULATION FOR ROBUST AND EFFICIENT INTEGRATION OF FINITE DIFFERENCES AND PHASE UNWRAPPING ON SPARSE MULTIDIMENSIONAL DOMAINS

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ABSTRACT
Phase unwrapping and integration of finite differences are key problems in several technical fields. In SAR interferometry and differential and persistent scatterers interferometry digital elevation models and displacement measurements can be obtained after unambiguously determining the phase values and reconstructing the mean velocities and elevations of the observed targets, which can be performed by integrating differential estimates of these quantities (finite differences between neighboring points). In this paper we propose a general formulation for robust and efficient integration of finite differences and phase unwrapping, which includes standard techniques methods as sub-cases. The proposed approach allows obtaining more reliable and accurate solutions by exploiting redundant differential estimates (not only between nearest neighboring points) and multi-dimensional information (e.g. multitemporal, multi-frequency, multi-baseline observations), or external data (e.g. GPS measurements). The proposed approach requires the solution of linear or quadratic programming problems, for which computationally efficient algorithms exist. The validation tests obtained on real SAR data confirm the validity of the method, which was integrated in our production chain and successfully used also in massive productions.

1. INTRODUCTION
Phase unwrapping and integration of finite differences are key problems in several technical fields. In SAR interferometry and differential and persistent scatterers interferometry [1], [2], [3] the quality of the derived products, digital elevation models and displacement measurements, is strongly dependent on the capability of unambiguously determining the phase values and reconstructing the mean velocities and elevations of the observed targets. In both cases a set of differential estimates (finite differences between neighboring points), generally not consistent, have to be integrated in order to reconstruct the values of the function of interest in a sparse multidimensional grid of points.

In the phase unwrapping problem finite difference estimates to be integrated are typically obtained by assuming that the differences of phases in neighboring points are smaller than half a cycle. In the elevation and velocity reconstruction problem in persistent scatterer interferometry, finite difference estimates to be integrated are found by spectral estimation techniques or maximizing the so-called temporal coherence. Therefore, the two problems can be formulated identically as the problem of reconstructing a function given preliminary estimates of its finite differences, which in general are not consistent, i.e. their integration depends on the integration path. The only difference in these two problems is in the allowed corrections to the preliminary estimates: in phase unwrapping the corrections should be discrete (integer multiples of $2\pi$), while in the general finite difference integration problems the corrections can take any values. This makes the phase unwrapping problem more difficult, with a lot of literature produced on this topic [4], [5], [6], [7], [8], [9], [10].

In SAR interferometry (including differential and persistent scatterer interferometry) often a stack of images acquired at different times, baselines or frequencies are available. Usually, those data are processed independently or at most working on one domain at once [11]. Some techniques for 3D phase unwrapping have been proposed [12], [13], but they need a preliminary data calibration to compensate different atmosphere and orbital phase contributions at different times in order to be effectively applied for multitemporal phase unwrapping.

We propose here a general formulation for integration of finite differences and phase unwrapping. The proposed approach includes standard phase unwrapping and finite difference integration techniques as special cases, but allows obtaining more robust to noise and outliers by exploiting redundant information, obtained working with differences between not only nearest neighboring pixels.

Moreover, the proposed formulation allows exploiting multi-dimensional information (e.g. multitemporal, multi-frequency, multi-baseline
Let \( f_i, \ i \in S \), be a function defined on the set \( S \) of (generally sparse) points on a multidimensional domain. Assume that preliminary estimates of the values of some linear combinations of \( \{ f_i \}_{i \in S} \) are known, i.e. that the set \( T \) of equations \( \sum_{j \in S} (a_{ij} f_j) - b_i = 0, \ i \in T \), hold with a certain degree of approximation and/or for most of cases. Consider then the problem of reconstructing the function \( f_i, \ i \in S \). In the cases of our interest the cardinality of \( T \) is greater than the cardinality of \( S \), and the preliminary estimates are not error free nor consistent together, i.e. the available information for the function reconstruction is redundant, but not exact, although unbiased.

The reconstruction problem can be formulated as the inversion of an overdetermined system of linear equations. The solution can be determined by minimizing, according to a given metric, the residuals \( \delta_i, \ i \in T \), of the different equations:

\[
\min_\delta \sum_{i \in T} c_i |\delta_i|^p
\]

subject to the constraints

\[
\sum_{j \in S} a_{ij} f_j - \delta_i = b_i, \quad i \in T.
\]

where the positive exponent \( p \) defines the selected metric and \( c_i \) are positive costs or weights chosen based on the reliability of the preliminary estimates. With \( p = 2 \) (\( L_2 \) norm) (1) and (2) define a quadratic programming (QP) problem. With \( p = 1 \) (\( L_1 \) norm), (1) and (2) reduce to a linear programming (LP) problem after the following change of variables (as already done in [8], [9]):

\[
\delta_i = \delta_i^+ - \delta_i^-, \quad \delta_i^+ \geq 0, \quad \delta_i^- \geq 0, \quad i \in S.
\]

Computationally efficient solving algorithms exist both for QP and LP problems. The \( L_2 \) metric is more appropriate when a normal distribution of the errors \( \delta_i \) is expected, whereas the \( L_1 \) norm guarantees solutions more robust to outliers, avoiding error spreading, and is therefore very useful for the applications examined in this paper. Moreover, we will see in the next section that the \( L_1 \) metric can guarantee in some cases integer solutions, a desired property for phase unwrapping.

### 2.2. Redundant finite difference integration and phase unwrapping

Phase unwrapping is the problem of reconstructing a function given its value modulo \( 2\pi \). Preliminary estimates of the unwrapped phase differences between neighboring pixels are usually found by assuming that their absolute values are smaller than \( \pi \) (see [1], [2] for SAR interferometry examples). Therefore phase unwrapping reduces to the problem of integrating finite differences, with possibly the constraint that the reconstructed function should differ by integer multiples of \( 2\pi \) from the original wrapped phase.

Finite difference integration problems are also the reconstructions of elevations and mean velocities of sparse coherent points or persistent scatterers (PS) in SAR interferometry, starting from their neighboring pixel differences estimates obtained by spectral estimation techniques or maximizing the so called temporal coherence [3].

In order to treat the finite difference integration problem it is convenient to define the graph \((S,A)\) whose nodes are the considered (generally sparse) points \( i \in S \) on a multidimensional domain (often two-dimensional in practical applications, but the formulation is general), and whose arcs \((i,j) \in A\) connect points \( i \in S, j \in S \), typically neighboring, but not necessarily only nearest neighbors (see Figure 1).

The finite difference integration problem consists in the reconstruction of the function \( f_i, \ i \in S \), from the preliminary estimates \( f'_{ij} \) of \( f_i - f_j, \ (i,j) \in A \). According to the general formulation (1), (2), the problem can be stated as:

\[
\min_\delta \sum_{i,j \in A} c_{ij} |\delta_{ij}|^p
\]

subject to the constraints

\[
f_i - f_j - \delta_{ij} = f'_{ij}, \quad (i,j) \in A.
\]
Assuming that the edge connectivity of the graph \( (S, A) \) is greater than one, i.e. that exist at least two distinct paths that connect any two nodes of the graph, then it can be easily demonstrated than the constraints (5) are equivalent to “irrationality” constraints. In fact, let \( C \) be a sequence of arcs forming a closed path, and \( G \) a set of independent closed paths that span the whole cycle-space. Then, by summing for each \( C \in G \) the equations in (5) that corresponds to arcs belonging to \( C \), the following set of equations is obtained:

\[
- \sum_{(i,j) \in C} \delta_{ij} = \sum_{(i,j) \in C} f'_{ij}, \quad C \in G .
\]

By solving (4) with the new constraints (6), the “minimum corrections” \( \delta_{ij} \), \( (i,j) \in A \), are obtained such that the integral of \( f'_{ij} + \delta_{ij} \) does not depend on the integration path and defines the function \( f_i \), \( i \in S \), up to an additive constant. In this sense, the problems defined by the objective function (4) with the constraints (5) or (6), respectively, are equivalent.

It is important to note that in the LP problem obtained using \( p = 1 \) (\( L_1 \) norm) and the change of variables (3), the matrix of constraints (5) is totally unimodular. Moreover, in the case of phase unwrapping, the constraints (6) can be written with only integer parameters. Using these two properties, it can be seen that the LP phase unwrapping solution differs by integer multiples of \( 2\pi \) from the original wrapped phase function.

When the graph \( (S, A) \) is planar, i.e. the graph can be represented in a two-dimensional plane without intersecting arcs (as for example the graph obtained by Delaunay triangulation), then the dual graph can be constructed. The dual graph is obtained by placing a node inside each elementary closed path (face) of the primal graph and a node (the root node) in the unbounded face outside the primal graph; moreover, an arc crossing each arc of the primal graph is considered to connect two nodes of the dual graph. As shown in [8], [9], [10], the LP integration problem on the primal graph is equivalent to the problem of finding the minimum cost flow on the dual network, for which solving algorithms exist that are computationally extremely efficient.

The main improvement of the finite difference integration (and phase unwrapping) approach defined in (4) and (5) or (6) with respect to previous techniques, is that the proposed formulation is valid for generic graphs, not only planar ones. We considered three-dimensional and in general multidimensional graphs, which are not planar, already in [12]. However, here we propose to use a graph with a very redundant number of arcs, i.e. connecting not only nearest neighboring points (see Figure 1), which makes possible to obtain a solution that is robust to outliers and noise.

We can call this method redundant finite difference integration and phase unwrapping algorithm.

### 2.3. Multitemporal phase unwrapping

The redundant finite difference integration and phase unwrapping method defined by (4) and (5) or (6) is valid for generally sparse data on multidimensional domains. However, the multidimensional space is considered isotropic, whereas the general approach stated in (1), (2) and (3) allows conceiving different formulations that better capture specific structures of the data to be processed.

In multitemporal SAR interferometry, a three-dimensional (3D) structure is obtained considering several two-dimensional (2D) SAR images acquired at different times. A key problem is to unwrap the phases of every image (with respect to a master one): unwrapping the phases at different times not independently, but jointly and consistently can make for better results. However, it is important to consider that the measured phases contain disturbing contributions due to propagation of the signal through the atmosphere, inaccurate knowledge of the acquisition positions (orbits), etc., which are almost identical in points close in space, but very different at different times. Therefore, no reliable estimation of the unwrapped phase differences between pixels at different times can be obtained, and the technique stated in (4) and (5) or (6) (or the less general proposed in [12]), cannot be applied as it were a real 3D phase unwrapping problem.

Assume that the set of sparse points \( S \) in a multidimensional domain has a multi-layer structure, i.e. \( S \) can be represented as \( S_1 \times S_2 \), where \( S_1 \) and \( S_2 \) are sets of sparse points in lower dimension domains, and let \( f_{ik} \) be a function on the set of sparse points.
\((i, k) \in S_1 \times S_2\), where \(i \in S_1\) and \(k \in S_2\) indicate the positions in the two domains. In the considered multitemporal phase unwrapping application \(S_1\) is the set of considered sparse pixel where the phase is meaningful at all times (i.e. the coherent points or the persistent scatterers) and \(S_2\) the set of available phase measurements at different times.

It is convenient in this case to consider two graphs: the graph \((S_1, A_1)\) whose nodes are the points \(i \in S_1\) on the first domain (two-dimensional space), and whose arcs \((i, j) \in A_1\) connect points \(i \in S_1\), \(j \in S_1\) (typically neighboring, but not necessarily only nearest neighbors), and an analogous graph \((S_2, A_2)\) on the second domain (one-dimensional time in the considered case).

Finally, let \(\hat{f}_{ijkl}^{pre}\) be preliminary estimates of \(f_{ik} - f_{il} - f_{jk} + f_{jl}\) \((i, j) \in A_1, (k, l) \in A_2\). These estimates can be obtained in the considered phase unwrapping problem by assuming that their absolute values are smaller than \(\pi\), which is true in most cases. In fact, in the considered differences between neighboring points, contributions slowly variable in space like atmospheric and orbital artifacts tend to cancel out. Moreover, the elevation contribution to the phase can be typically removed prior to phase unwrapping using an available digital elevation model [1], [2], and spectral estimation techniques [3]). Then, according to the general formulation (1), (2), the problem of reconstructing the function \(f_{ik}, (i, k) \in S_1 \times S_2\), can be stated as:

\[
\min_{\delta_{\mu i}} \sum_{(i,j) \in A_1, (k,l) \in A_2} c_{\mu i} |\delta_{\mu i}|^p
\]

subject to the constraints

\[
f_{ik} - f_{il} - f_{jk} + f_{jl} - \delta_{\mu i} = \hat{f}_{ijkl}^{pre}, \quad (i, j) \in A_1, (k, l) \in A_2.
\]

As already discussed, the exponent \(p\) defines the metric, and computationally efficient solving algorithms exist for \(p = 2\) (quadratic programming), and \(p = 1\) (linear programming) after using the change of variables (3). The linear programming solution can guarantee in some conditions that the reconstructed unwrapped phases differ by integer multiples of \(2\pi\) from the original wrapped phases.

2.4. Multi-baseline and multi-frequency problems

Like multi-temporal phase unwrapping, several other problems are characterized by non isotropic multidimensional (multi-layer) data structures, which can be captured by the general approach stated in (1), (2) and (3). Interesting problems are those arising when multiple observations are done in SAR interferometry from slightly different view angles (multi-baseline interferometry) or at different frequencies (wide-band interferometry). In the case of multi-baseline
acquisitions the scene is observed ideally at the same
time, or at close times assuming that changes and
displacements did not happen between the acquisition
times. In the case of multi-frequency observations, two
sets of acquisitions are considered, normally taken from
two slightly different view angles and/or at two close
times, and for each set a number of measurements at
distinct frequencies (the same for the two sets) are
performed.

The multi-frequency and multi-baseline SAR
interferometry problems can be formulated in the same
way as the multitemporal phase unwrapping discussed
in section 2.3, with $S_2$ being the set of available
observations. However, in these cases an additional
property holds: the phase differences $f_i - f_l$, $i \in S_1$,
$k, l \in A_2$, are proportional to the frequency or baseline
values associated to the observation pairs $(k, l)$,
respectively. Note, however, that in the case of multi-
frequency acquisitions the set $A_2$ contains only arcs $(k, l)$
that connect measurements at the same frequency.

Using this property, a unique function $g_i, i \in S_1$, can be
defined such that

$$f_i - f_l = \alpha_k g_i, \quad i \in S_1, (k, l) \in A_2,$$

with $\alpha_k$ being a scaling coefficient proportional to the
frequency or the baseline value associated to the pair of
observations $(k, l) \in A_2$. By substituting (9) in (7), (8),

subject to the constraints

$$\alpha_k (g_i - g_j) - \delta_{ijkl} = f'_{ijkl}, \quad (i, j) \in A_1, (k, l) \in A_2,$$

where, as in (7), (8), $p$ defines the metric, and $f'_{ijkl}$ are
preliminary estimates of $f_i - f_l - f_k + f_j$, $(i, j) \in A_1$,
$(k, l) \in A_2$. These preliminary estimates can be obtained
as usual assuming that their absolute values are smaller
than $\pi$, which is mostly true when the arcs $(i, j) \in A_1$
connect points spatially neighboring (in this case, however,
the elevation contribution to the phase is typically the object of the search and cannot be removed
in advance). As for the metric, it is obviously not
anymore true that the reconstructed phases differ by
integer multiples of $2\pi$ from the original wrapped phases.

2.5. Other applications and use of external
information

The general formulation stated in (1), (2) and (3) is
suitable for many other applications in addition to those
shown in sections 2.2, 2.3 and 2.4. With the proposed
formulation, any kind of linear constraint between the

above figure shows the results of the consistency check (12)
performed for the Delaunay triangulation phase unwrapping
using three

Envisat acquisitions at the dates (a) 2004/05/26, (b) 2004/06/30,
(c) 2004/08/04) over Abruzzo, Italy. The

inconsistencies, i.e. the points where (12) is not verified are
shown in colors different from green. More than 7% of points
are inconsistently unwrapped.

The above figure shows the results of the same consistency check of Figure 4 for the redundant phase unwrapping method described in
section 2.2. Less than 2% of points are inconsistently
unwrapped.
values of the function to be reconstructed can be considered. It is important to note that the constraints must be linear with respect to the values of the function to be reconstructed, but the coefficients can describe nonlinear behaviors with respect to the domains in which the function is defined, for example with respect to the time.

Among the possible applications, it is worth mentioning that the proposed formulation allows straightforward including information coming from external sources, like GPS or other in-situ measurements, in SAR interferometry reconstruction problems. In fact, if the function to be reconstructed is known in a set of points \( i \in S \subseteq \mathcal{S} \), it is very easy to add constraints expressing this information to the minimization problem to be solved.

3. EXPERIMENTAL RESULTS

The proposed method was validated and integrated in our production chain, with successful use also in massive productions. In this section we report, as an explicative example, one test relative to the most difficult finite difference integration problem we have considered, i.e. phase unwrapping. We used two stacks of Envisat SAR images over Campania and Abruzzo regions, Italy. We compared different phase unwrapping strategies on these data: based on nearest neighbors defined by the Delaunay triangulation \([10]\), or on redundant differences between nearby points instead that only nearest neighbors (described in section 2.2).

In order to determine the quality of the results, we applied a consistency check. Let \( a, b, c \) be three SAR images and let \( \phi_{ab}, \phi_{bc}, \) and \( \phi_{ca} \) be the unwrapped interferometric phase functions (defined on a sparse set of points, the persistent scatterers) associated to the interferograms \( (a, b), (b, c) \) and \( (a, c) \), respectively. To be consistent these unwrapped phases should obviously verify in each point the following relation:

\[
\phi_{ab} + \phi_{bc} - \phi_{ca} = 0. \tag{12}
\]

Figure 2 and Figure 3 show the results of the consistency check (12) performed using three Envisat acquisitions at the dates \((a)\) 2004/05/26, \((b)\) 2004/06/30, \((c)\) 2004/08/04) over the Campania, Italy, for the Delaunay triangulation and the redundant phase unwrapping, respectively. The results of the same tests performed on Envisat SAR data acquired over Abruzzo, Italy, at the dates \((a)\) 2004/05/26, \((b)\) 2004/06/30, \((c)\) 2004/08/04, are shown in Figure 4 and Figure 5.

The performed test indicate that the with the redundant arc phase unwrapping method phase unwrapping errors are drastically reduced with the respect to the standard technique in which only Delaunay triangulation arcs are considered.

The multitemporal phase unwrapping method described in section 2.3 verifies by definition the consistency check (12). In fact, the consistency is one of the conditions enforced in this case, to help finding a better solution.

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REFERENCES