Multiscale Modelling of GOCE Data Products

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Abstract

The lecture aims at describing the ZOOM IN proposal by the Fraunhofer Institute for Industrial Mathematics (ITWM, Kaiserslautern) in cooperation with the Geomathematics Group of the University of Kaiserslautern within the framework of the German geotechnology programme 'Observation of the System Earth from Space'.

The research will be concentrated on the areas gravitational field, magnetic field, and density variations. The application to a high–precision geoid based on GOCE data products in combination with local airborne and/or terrestrial observations will be accounted for. This area is dealt in parallel with multiresolution analysis of the magnetic field by (pre–)Maxwell wavelets, and the interior density field by use of multiscale gravimetry in combination with seismic records.

ZOOM IN is intended to offer multiscale models within the desired scales from the global observation of our planet from space up to regional dimensions. Essential tools are new mathematical methods for evaluating satellite data, based on multiscale analysis by wavelets. The multiscale procedures shall be further developed systematically and implemented as a homogeneous software structure. Thus, the user has an instrument on hand that is classified according to wavelength, frequency, space and time, which results in a better understanding of the interrelations and interactions and a scale–specific observation of the system Earth.

Contents of the Lecture (Key–Word–Representation)

(Conventional) Spherical Harmonic Approximation

$$F(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \int_{\Omega} F(y) Y_{n,k}(y) \; d\omega(y) \quad Y_{n,k}(x)$$

$\{Y_{n,k}\}_{\substack{n=0,1,\ldots \\ k=1,\ldots,2n+1}}$: *spherical harmonics* on the (unit) sphere Ω

Character of the spherical harmonics: ideal frequency localization, no space localization.

Comments:

(i) Fourier methods in terms of spherical harmonics are successful at picking out frequencies from a (spherical) signal, but they are incapable of dealing properly with data changing on small spatial scales.

(ii) The space evolution of the frequencies is not reflected in the Fourier transform in terms of non–space–localizing spherical harmonics.

(Formal) Convolution Against the Dirac Function(al)

$$F(x) = \int_{\Omega} \delta(x,y) F(y) \; d\omega(x)$$

$\delta(\cdot,\cdot)$: *Dirac functional* on the (unit) sphere Ω

$$\begin{aligned} \delta(x,y) &= \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \delta^{\wedge}(n) Y_{n,k}(x) Y_{n,k}(y) \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \delta^{\wedge}(n) P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \quad x, y \in \Omega \end{aligned}$$

with P_n Legendre polynomial of degree n and

$$\delta^{\wedge}(n) = 1, \quad n = 0, 1, \ldots \; .$$

Character of Dirac function(als): no frequency localization, ideal space localization

Multiscale philosophy: Medio tutissimus ibis.

Linear Approximation by Scaling Functions

$$F(x) = \lim_{\rho \to 0} \int_{\Omega} \Phi_{\rho}(x,y) F(y) \; d\omega(y)$$

$\Phi_{\rho}(\cdot,\cdot)$: *scaling function* (see W. Freeden et al. (1998))

$$\Phi_{\rho}(x,y) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \Phi_{\rho}^{\wedge}(n) Y_{n,k}(x) Y_{n,k}(y), \quad x, y \in \Omega$$

with

$$\Phi_\rho(x,y) \underset{\rho\to 0}{\longrightarrow} \delta(x,y) \qquad\qquad \Phi_\rho^\wedge(n) \underset{\rho\to 0}{\longrightarrow} 1.$$

(space condition) (frequency condition)

Bilinear Approximation by Scaling Functions

$$\begin{aligned} F(x) &= \lim_{\rho\to 0} \int_\Omega \Phi_\rho^{(2)}(x,y) F(y)\, d\omega(y) \\ &= \lim_{\rho\to 0} \int_\Omega \Phi_\rho(x,z) \int_\Omega \Phi_\rho(z,y) F(y)\, d\omega(y)\, d\omega(z) \end{aligned}$$

with

$$\Phi_\rho^{(2)}(x,y) = \int_\Omega \Phi_\rho(x,z)\Phi_\rho(z,y)\, d\omega(z)$$

and

$$\Phi_\rho^{(2)}(x,y) \underset{\rho\to 0}{\longrightarrow} \delta(x,y) \qquad\qquad (\Phi_\rho^{(2)})^\wedge(n) = (\Phi_\rho^\wedge(n))^2 \underset{\rho\to 0}{\longrightarrow} 1.$$

(space condition) (frequency condition)

Character of scaling functions: occurence of all stages of frequency as well as space localization

uncertainty principle: (see W. Freeden (1998, 1999), W. Freeden, V. Michel (1999))

ideal frequency localization — no frequency localization
no space localization — ideal space localization

$\longleftarrow \cdots\cdots \longleftrightarrow \cdots\cdots \longrightarrow$

bandlimited/non–bandlimited

spherical harmonics	kernel functions		Dirac function(als)
$K^\wedge(n) = \delta_{n,l}$ for $l \geq 0$	$K^\wedge(n) = 0$ for all $n \geq N$	$K^\wedge(n) \neq 0$ for infinite n	$K^\wedge(n) = 1$ for all n

(kernel functions (radial basis function on the sphere Ω))

$$K(x,y) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} K^\wedge(n) Y_{n,k}(x) Y_{n,k}(y), \quad x,y \in \Omega$$

Comment:
Multiscale methods automatically adapt the amount of localization in space and in frequency. Only a narrow space–window is needed to examine high–frequency content, but a wide space–window is allowed when investigating low frequency phenomena.

Scale Discretization:

$$(\rho_j)_{j=0,1,\ldots}: \text{ sequence with } \lim_{j\to\infty} \rho_j = 0.$$

Linear Approximation by Wavelet Functions

$$\begin{aligned}\int_\Omega \Psi_{\rho_j}(x,y)F(y)\ d\omega(y) &= \int_\Omega \Phi_{\rho_{j+1}}(x,y)F(y)\ d\omega(y) \\ &- \int_\Omega \Phi_{\rho_j}(x,y)F(y)\ d\omega(y)\end{aligned}$$

with

$$\Psi_{\rho_j}(x,y) = \Phi_{\rho_{j+1}}(x,y) - \Phi_{\rho_j}(x,y) \qquad \Psi^\wedge_{\rho_j}(n) = \Phi^\wedge_{\rho_{j+1}}(n) - \Phi^\wedge_{\rho_j}(n).$$

(space condition) (frequency condition)

Reconstruction formula

$$F(x) = \underbrace{\int_\Omega \Phi_{\rho_0}(x,y)F(y)\ d\omega(y)}_{\text{low pass filter}} + \lim_{J\to\infty} \sum_{j=0}^{J-1} \underbrace{\int_\Omega \Psi_{\rho_j}(x,y)F(y)\ d\omega(y)}_{\text{band pass filter}}$$

Bilinear Approximation by Wavelet Functions

$$\Psi^{(2)}_{\rho_j}(x,y) = \Phi^{(2)}_{\rho_{j+1}}(x,y) - \Phi^{(2)}_{\rho_j}(x,y) \qquad (\Psi^\wedge_{\rho_j}(n))^2 = (\Phi^\wedge_{\rho_{j+1}}(n))^2 - (\Phi^\wedge_{\rho_j}(n))^2.$$

(space condition) (frequency condition)

$$F(x) = \underbrace{\int_\Omega \Phi^{(2)}_{\rho_0}(x,y)F(y)\ d\omega y}_{\text{low pass filter}} + \lim_{J\to\infty} \sum_{j=0}^{J-1} \underbrace{\int_\Omega \Psi_{\rho_j}(x,y)(WT)_j(F)(y)\ d\omega(y)}_{\text{band pass filter}}$$

Wavelet transform

$$(WT)_{\rho_j}(F)(y) = \int_\Omega \Psi_{\rho_j}(y,z)F(z)\ d\omega(z)$$

Multiresolution Analysis

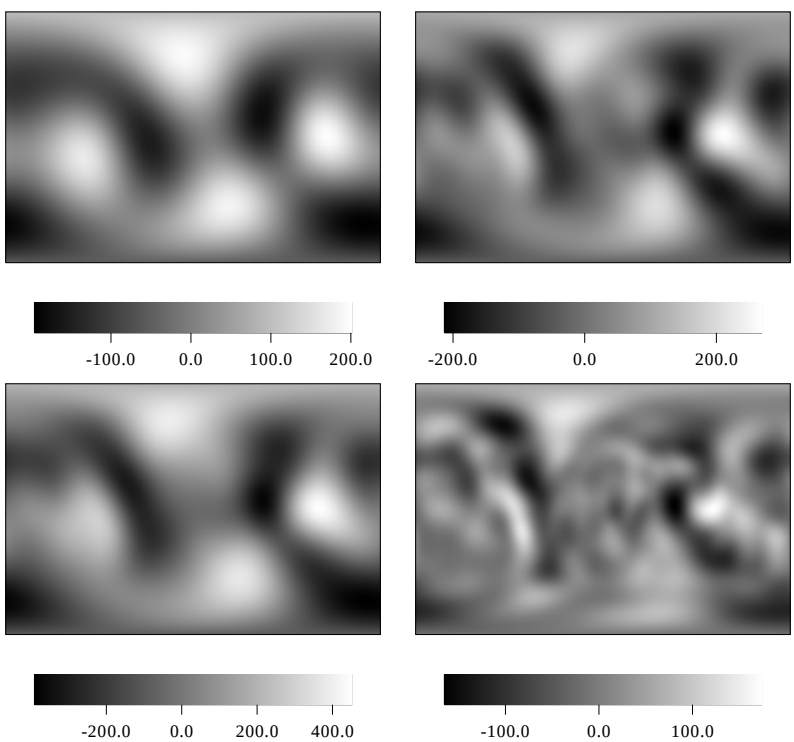
$$
\begin{array}{ccccccccc}
L_{\rho_0}(F) & & L_{\rho_1}(F) & & L_{\rho_2}(F) & \dots & \underset{j\to\infty}{\longrightarrow} & F \\
\cap & & \cap & & \cap & & & \cap \\
\mathcal{V}_{\rho_0} & \subset & \mathcal{V}_{\rho_1} & \subset & \mathcal{V}_{\rho_2} & \dots & = & L^2(\Omega) \\
\hline
\mathcal{V}_{\rho_0}+ & W_{\rho_0} & + & W_{\rho_1} & + \; W_{\rho_2} & \dots & = & L^2(\Omega) \\
\cup & \cup & & \cup & \cup & & & \cup \\
L_{\rho_0}(F)+ & B_{\rho_0}(F) & + & B_{\rho_1}(F) & + B_{\rho_2}(F)+ & \dots & = & F
\end{array}
$$

V_{ρ_j}: **scale spaces**,

$$L_{\rho_j}(F) = \int_\Omega \Phi_{\rho_j}(\cdot, y) F(y)\, d\omega(y) \in V_{\rho_j}, \; j = 0, 1, \dots$$

W_{ρ_j}: **detail spaces**

$$B_{\rho_j}(F) = \int_\Omega \Psi_{\rho_j}(\cdot, y) F(y)\, d\omega(y) \in W_{\rho_j} \;, \quad j = 0, 1, \dots .$$



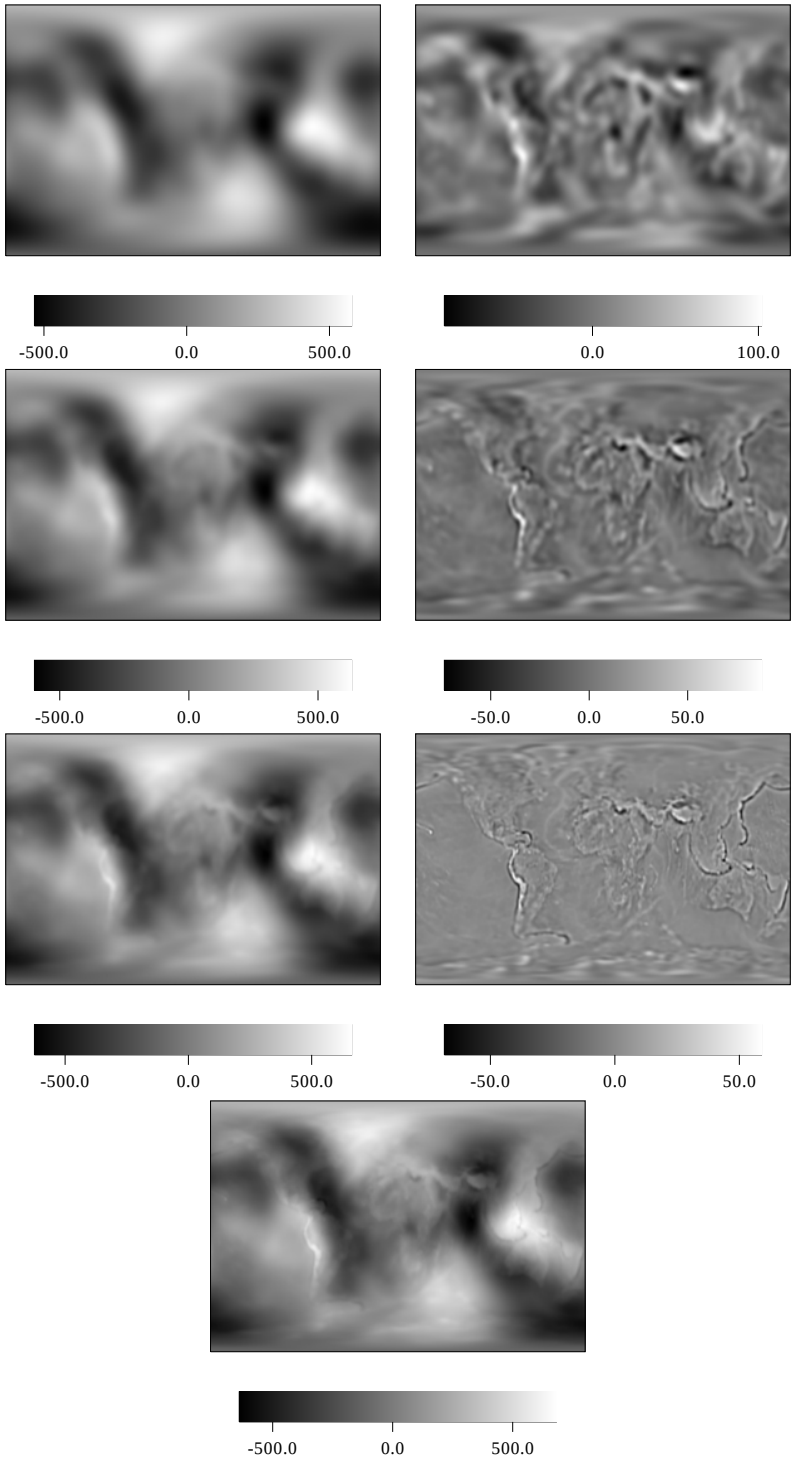


Figure 1: Multiresolution of the EGM96–Modell (see Freeden, Glockner, Litzenberger (1999)): band–limited scale space (left column) and detail space (right column), reconstruction of the model at scales 3 (top) to 8 (bottom)

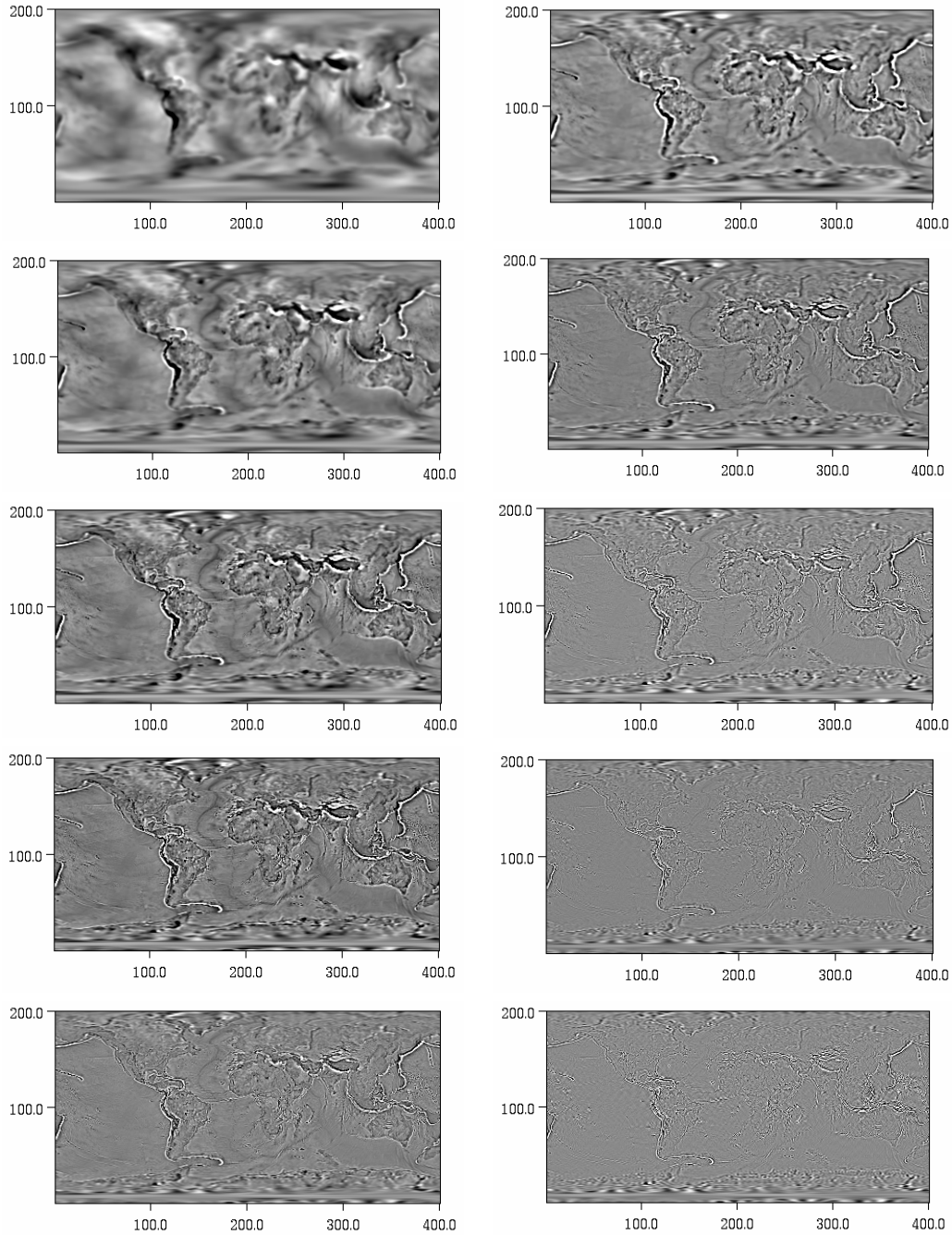


Figure 2: Multiresolution of the Harmonic Density Variations from EGM96–Modell (see V. Michel (1999)): non–band–limited scale space (left column) and detail space (right column) reconstruction of the harmonic density at scales 4 (top) to 8 (bottom)

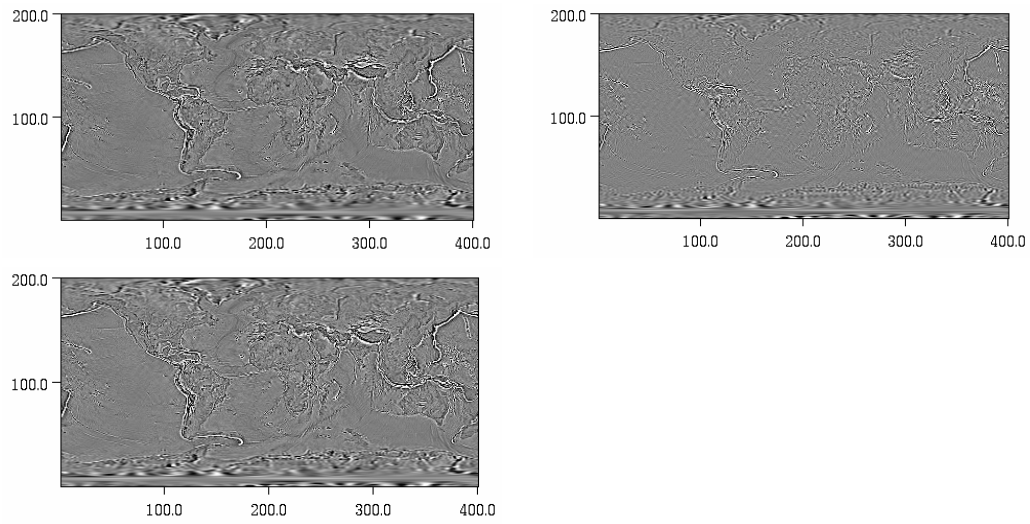


Figure 3: Multiresolution of the Harmonic Density Variations from EGM96–Modell (see V. Michel (1999)): non–band–limited scale space (left column) and detail space (right column) reconstruction of the harmonic density at scales 9 (top) and 10 (bottom)

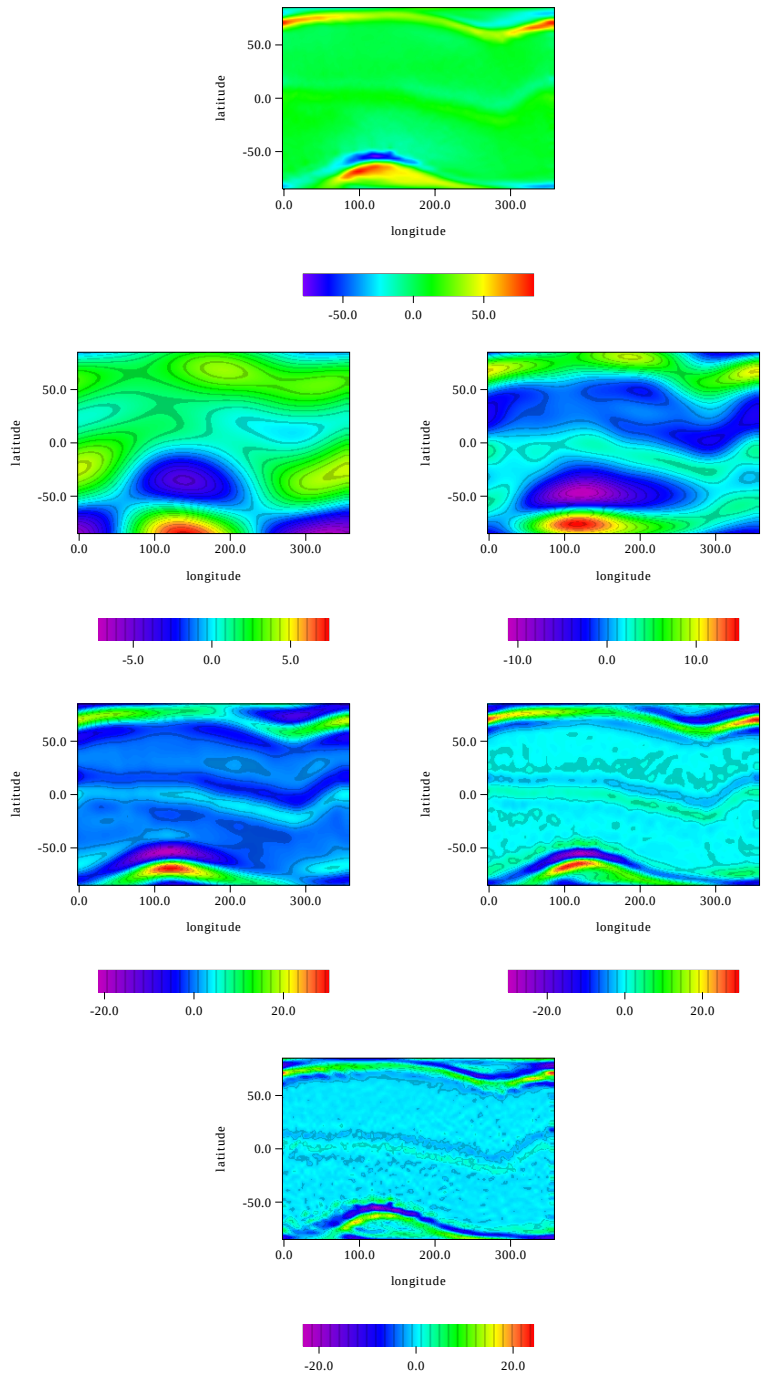


Figure 4: Partial reconstructions of MAGSAT toroidal morning Y component of Earth's magnetic field (from detail space 2 to detail space 6, the top level picture shows the corresponding complete toroidal Y component), see M. Bayer et al. (2001), W. Freeden, T. Maier (2001)

Approximate Integration

$$
\begin{aligned}
L_{\rho_j}(F) &= \int_\Omega \Phi_{\rho_j}(x,y)F(y)\,d\omega(y) \cong \sum_{i=1}^{N_j} c_i^{N_j}\Phi_{\rho_j}(x,y_i^{N_j})F(y_i^{N_j}),\\
B_{\rho_j}(F) &= \int_\Omega \Psi_{\rho_j}(x,y)F(y)\,d\omega(y) \cong \sum_{i=1}^{N_j} c_i^{N_j}\Psi_{\rho_j}(x,y_i^{N_j})F(y_i^{N_j}).
\end{aligned}
$$

Example: Weyl's method of equidistribution i.e. $c_i^{N_j} = \frac{4\pi}{N_j}$ (see W. Freeden et al. (1998)).

Fast Wavelet Transform (Error Free Data)

(1) J sufficiently large:

$$
\begin{aligned}
J : F(x) \cong F_{\rho_J}(x) &= \int_\Omega \Phi_{\rho_J}(x,y)F(y)\,d\omega(y)\\
&\cong \sum_{i=1}^{N_J} a_i^{N_J}\Phi_{\rho_J}(y_i^{N_J},x)
\end{aligned}
$$

('**read in**'–process)

with

$$
a^{N_J} = (a_1^{N_J},\ldots,a_{N_J}^{N_J})^T, \qquad a_i^{N_J} = c_i^{N_J}F(y_i^{N_J})
$$

(2) $a_i^{N_j}$ recursively calculable by $a_i^{N_{j+1}}$

(3) $j = 0,\ldots,J-1 : F_{\rho_j}(x) \cong \sum_{i=1}^{N_j} a_i^{N_j}\Phi_{\rho_j}(y_i^{N_j},x)$

('**recursion**'–process)

$$
\boxed{
\begin{array}{ccccc}
a_i^{N_j} & \cong & c_i^{N_j} & \sum_{l=1}^{N_{j+1}} K_{V_{\rho_j}}(y_i^{N_j},y_l^{N_{j+1}}) & a_l^{N_{j+1}}\\
 & & \uparrow & \uparrow & \uparrow\\
 & & \text{integration-weights} & \text{reprokernel of } V_{\rho_j} & \text{known from data}
\end{array}
}
$$

decomposition scheme:

$$
\begin{array}{ccccccccc}
F & \longrightarrow & a^{N_J} & \longrightarrow & a^{N_{J-1}} & \longrightarrow & \ldots & \longrightarrow & a^{N_0}\\
 & & \downarrow & & \downarrow & & & & \downarrow\\
 & & L_{\rho_j}(F) & & L_{\rho_{j-1}}(F) & & & & L_{\rho_0}(F)
\end{array}
$$

reconstruction scheme:

$$
\begin{array}{ccccccccccc}
a^{N_0} & & & & a^{N_1} & & & & a^{N_2} & & \\
\downarrow & & & & \downarrow & & & & \downarrow & & \\
B_{\rho_0}(F) & \searrow & & & B_{\rho_1}(F) & \searrow & & & B_{\rho_2}(F) & \searrow & \\
L_{\rho_0}(F) & \longrightarrow & + & \longrightarrow & L_{\rho_1}(F) & \longrightarrow & + & \longrightarrow & L_{\rho_2}(F) & \longrightarrow & + \quad \ldots
\end{array}
$$

Spectral Signal Variances

signal energy:

$$\|F\|^2_{L^2(\Omega)} = \int_\Omega |F(y)|^2 \, d\omega(y)$$

Parseval identity:

$$\|F\|^2_{L^2(\Omega)} = \sum_{\underset{\text{degree}}{n}} \sum_{\underset{\text{order}}{k}} (F^\wedge(n,k))^2$$

Signal degree and order variances:

$$\mathrm{var}_{n,k}(F) = \left(\int_\Omega F(y) Y_{n,k}(y) \, d\omega(y) \right)^2 = (F^\wedge(n,k))^2$$

Signal degree variances:

$$\mathrm{var}_n(F) = \sum_{k=1}^{2n+1} \mathrm{var}_{n,k}(F)$$

Multiscale Signal Variances

signal energy:

$$\|F\|^2_{L^2(\Omega)} = \int_\Omega |F(y)|^2 \, d\omega(y)$$

Parseval identity:

$$\|F\|^2_{L^2(\Omega)} = \sum_{\underset{\text{scale}}{j}} \int_{\underset{\text{space}}{\Omega}} \left(\overbrace{\int_\Omega F(z) \Psi_{\rho_j}(y,z) \, d\omega(z)}^{\text{wavelet coefficients}} \right)^2 d\omega(y)$$

Signal scale and space variances:

$$\mathrm{var}_{j,y}(F) = \left(\int_\Omega F(z) \Psi_{\rho_j}(y,z) \; d\omega(z) \right)^2$$

Signal scale variances:

$$\mathrm{var}_j(F) = \int_\Omega \mathrm{var}_{j;y}(F) \; d\omega(y)$$

(see W. Freeden et al. (2001), W. Freeden, T. Maier (2001)).

Error Affected Data

$$\tilde{F} = F + \tilde{\varepsilon}, \quad \tilde{\varepsilon} : \text{observation noise.}$$

Covariance

$$E\left[\tilde{\varepsilon}(x), \tilde{\varepsilon}(y)\right] = K(x,y)$$

with

$$K(x,y) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} K^\wedge(n,k) Y_{n,k}(x) Y_{n,k}(y), \quad x,y \in \Omega \; .$$

spectral degree and order covariances

$$\mathrm{cov}_{n,k}(K) \quad = \quad \underset{\substack{\nearrow \quad \nwarrow \\ \text{degree} \quad \text{order}}}{K^\wedge(n,k)} = \int_\Omega \int_\Omega K(x,z) Y_{n,k}(y) Y_{n,k}(z) \; d\omega(x) \; d\omega(z)$$

spectral degree covariances

$$\mathrm{cov}_n(K) = \sum_{k=1}^{2n+1} K^\wedge(n,k)$$

multiscale scale and space covariances

$$\underset{\substack{\nearrow \quad \nwarrow \\ \text{scale} \quad \text{space}}}{\mathrm{cov}_{j,y}(K)} \quad = \quad \int_\Omega \int_\Omega K(x,z) \Psi_{\rho_j}(x,y) \Psi_{\rho_j}(z,y) \; d\omega(x) \; d\omega(z)$$

multiscale scale covariances

$$\mathrm{cov}_j(K) = \int_\Omega \mathrm{cov}_{j;y}(K) \; d\omega(y)$$

Spectral Signal–to–Noise (Hard) Thresholding

signal dominates noise

$$\mathrm{var}_{n,k}(\tilde{F}) \geq \mathrm{cov}_{n,k}(K), \qquad (n,k) \in \mathcal{N}_{\mathrm{res}}$$

noise dominates signal

$$\mathrm{var}_{n,k}(\tilde{F}) < \mathrm{cov}_{n,k}(K), \qquad (n,k) \notin \mathcal{N}_{\mathrm{res}}$$

$\mathcal{N}_{\mathrm{res}}$: degree and order resolution set.

Multiscale Signal–to–Noise (Hard) Thresholding
signal dominates noise

$$\mathrm{var}_{j;y}(\tilde{F}) \geq \mathrm{cov}_{j;y}(K), \qquad (j,y) \in \mathcal{Z}_{\mathrm{res}}$$

noise dominates signal

$$\mathrm{var}_{j;y}(\tilde{F}) < \mathrm{cov}_{j;y}(K), \qquad (j,y) \notin \mathcal{Z}_{\mathrm{res}}$$

$\mathcal{Z}_{\mathrm{res}}$: scale and space resolution set.

(Hard) Thresholding via Fast Wavelet Transform

$$\begin{aligned}
\tilde{a}_i^{N_j} &= \overbrace{c_i^{N_j}}^{\text{integration-weight}} \sum_{l=1}^{N_{j+1}} \overbrace{K_{V_{\rho_j}}(y_i^{N_j}, y_l^{N_{j+1}})}^{\text{Reprokernel of } V_{\rho_j}} a_l^{N_{j+1}} \text{(recursion)} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j = 0, \ldots, J-1) \\
\tilde{a}_i^{N_j} &= c_i^{N_J} \qquad\qquad \tilde{F}(y_i^{N_J}) \qquad\qquad \text{(read in)}
\end{aligned}$$

$$
\begin{aligned}
\left(\tilde{a}_i^{N_j}\right)^2 &= \left(c_i^{N_j}\right)^2 \left(\int_\Omega K_{V_{\rho_j}}(y, y_i^{N_j}) \overbrace{\tilde{F}(y)}^{\substack{\text{error–affected}\\ \text{data}}} \, d\omega(y) \right)^2 \\
&\geq \left(c_i^{N_j}\right)^2 \int_\Omega \int_\Omega K_{\rho_j}(z, y_i^{N_j}) K_{\rho_j}(w, y_i^{N_j}) \underbrace{K(z,w)}_{\substack{\text{covariance}\\ \text{kernel}}} \, d\omega(z) \, d\omega(w)
\end{aligned}
$$

$$
\begin{aligned}
\left(\tilde{a}_i^{N_j}\right)^2 &= \left(c_i^{N_j}\right)^2 \left(\int_\Omega K_{V_{\rho_j}}(y, y_i^{N_j}) \overbrace{\tilde{F}(y)}^{\substack{\text{error–affected}\\ \text{data}}} \, d\omega(y) \right)^2 \\
&< \left(c_i^{N_j}\right)^2 \int_\Omega \int_\Omega K_{\rho_j}(z, y_i^{N_j}) K_{\rho_j}(w, y_i^{N_j}) \underbrace{K(z,w)}_{\substack{\text{covariance}\\ \text{kernel}}} \, d\omega(z) \, d\omega(w)
\end{aligned}
$$

hard thresholding: 'keep' or 'kill' procedure

Consequences for GOCE Data Modelling

- regularization by multisresolution
  (see W. Freeden et al. (1997), W. Freeden, F. Schneider (1998))
- availability of a stop strategy
  (see W. Freeden, S. Pereverzev (2001), W. Freeden et al. (2001))
- no problems with gaps
- local improvement in global model
  (see W. Freeden et al. (1998))
- fast wavelet transform
  (see W. Freeden (1999), W. Freeden et al. (2001))
- adaptive use of millions of data, no linear system high accuracy resolution, no numerical instability
  (see W. Freeden (1999), F. Schneider (1997), W. Freeden, F. Schneider (1998), W. Freeden, O. Glockner, R. Litzenberger (2000))
- denoising
  (see W. Freeden et al. (1998), W. Freeden et al. (2001), W. Freeden, T. Maier (2001))
- data compression
  (see W. Freeden et al. (1998B))

- validation on local domains (first attempts in W. Freeden, F. Schneider (1998A))

Multiscale Philosophy of the Geomathematics Group:

approx. method	**Fourier**		**splines/wavelets**	**wavelets**
	orthogonal approximation		non–orthogonal approximation	
approx. structure	**polynomials**		bandlimited/non–bandlimited **kernels**	
	zooming–out		zooming–in	
localization	**increasing frequency localization**, **decreasing frequency localization** **decreasing space localization**, **increasing space localization**			
	increasing correlation		decreasing correlation	
data structure	**equidistributed**	weakly	**irregular**	strongly
	linear system	linear system/num. integ.		numerical integration
resolution	**long**	**medium**		**short**
		wavelengths		

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