Upward continuation of satellite altimeter data for GOCE validation

Sebera J., Bosch W., Bouman J., Bezděk A., Klokočník J., Kostelecký J.

Czech Technical University in Prague
Deutsches Geodätisches Forschungsinstitut, Germany
Astronomical Institute of the Academy of Sciences of the Czech Republic
Research Institute of Geodesy, Topography and Cartography, Czech Republic

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   ■ Satellite altimetry

2 Upward experiment
   ■ Global approximation
   ■ Input signal

3 Numerical comparisons
   ■ Upwarded 0th derivative: $T$
   ■ Upwarded 1st derivative: $T_r, T_z$
   ■ Upwarded 2nd derivative: $T_{rr}, T_{zz}$

4 Concluding remarks
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Roughly on satellite altimetry

- SA gives the geoid undulation since it holds \( MSS = DOT + N \) (MSS-mean sea surface, DOT-dynamic ocean topography).
- Having DOT at hand we can determine disturbing/anomalous potential from the Bruns formula \( T = N\gamma \).

SA - principle (from AVISO-CNES)

Some assumptions
- DOT is also unknown ⇒ not solved here
- SA coverage isn’t global ⇒ not considered
- ⇒ \( T \) globally is our input for the base functions experiments
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Numerical test of the approximation

1. Start with a signal on the geoid - $T$
2. Use both kinds of approximation $\Rightarrow \{C_{nm}^e, S_{nm}^e\}$ and $\{C_{nm}^s, S_{nm}^s\}$
3. Map both sets by upward operators onto the potential functionals at satellite altitude and compare EHS and SHS

Grid settings

- Regular grid on sphere (geocentric co-latitude $\theta$) is not regular on the ellipsoid (reduced co-latitude $\vartheta$)
- Trade-off $\Rightarrow$ mixture of both
- For SHA $P \in \{r, \theta + \vartheta(\vartheta), \lambda\}$
- For EHA $P \in \{u, \vartheta(\theta) + \vartheta, \lambda\}$

Harmonic analysis

- $\Rightarrow$ "Semi-regular" grid $f = (2N_{max} - 1, 2N_{max})$
- $\Rightarrow$ WLS solution for blocks used
- $\Rightarrow$ Latitudinal weights $W_i(\theta) = 2 \frac{\sin \theta_i}{\sum_{i=1}^{2N-1} \sin \theta_i}$
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Spherical harmonics

very well represent a functional \( f = f(r, \theta, \lambda) \) on Earth \( \sim \) spherical approximation

\[
T^s = \frac{GM}{a} \sum_{n,m} \left( \frac{a}{r} \right)^{n+1} (C_{nm}^s \cos m\lambda + S_{nm}^s \sin m\lambda) P_{nm}(\cos \theta)
\]  

(1)

Ellipsoidal harmonics

are much closer to Earth’s geometry, functional \( f = f(u, \vartheta, \lambda) \)

\[
T^e = \frac{GM}{a} \sum_{n,m} \frac{Q_{nm}(\frac{u}{E})}{Q_{nm}(\frac{b}{E})} (C_{nm}^e \cos m\lambda + S_{nm}^e \sin m\lambda) P_{nm}(\cos \vartheta)
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(2)

or with Jekeli’s renormalization \( S_{nm}(\frac{u}{E})/S_{nm}(\frac{b}{E}) = Q_{nm}(\frac{u}{E})/Q_{nm}(\frac{b}{E}) \)

”Normal” derivatives (\( z \) axis in LNOF)

\[
T^s_r = \frac{\partial V^s}{\partial r} \approx \frac{\partial V^e}{\partial \bar{z}} = T^e_{\bar{z}}
\]

\[
T^s_{rr} = \frac{\partial^2 V^s}{\partial r^2} \approx \frac{\partial^2 V^e}{\partial \bar{z}^2} = T^e_{\bar{z}\bar{z}}
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Spherical and ellipsoidal harmonic analysis of $T$ on the geoid

- Disturbing potential on the geoid, ITG03 model, $N_{max} = 180$
- Ellipsoidal and spherical analysis $N_{max} = 180$

$T^s$ from SHS, $[m^2 \cdot s^{-2}]$
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Degree variances of the derived coefficients
Synthesis of $T^s, T^e, T_{conv}^s$, $u = b + 255$ km, $N_{max} = 180$

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$T^e$ from EHS, $[m^2 \cdot s^{-2}]$
Synthesis of $T^s$, $T^e$, $T_{conv}^s$, $u = b + 255$ km, $N_{max} = 180$

$T^s - T^e$, RMS = $1.74 \text{ m}^2 \cdot \text{s}^{-2}$
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$T_{conv}^s - T^e$, RMS = 0.41 $m^2 \cdot s^{-2}$. 

Sebera et al. (CTU, DGFI, ASI, VUGTK)
Synthesis of $T_r, T_z$, $u = b + 255$ km, $N_{max} = 180$

$T_r^s$ from SHS, $[m \cdot s^{-2}]$
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$T_{r,\text{conv}} - T_{z}^e$, RMS = 0.014 mGal
Synthesis of $T_{rr}$, $T_{zz}$ at $u = b + 255$ km

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Concluding remarks

We have compared three sets of coefficients coming from one input (2x spherical and 1x ellipsoidal) via harmonic synthesis on the $u = b + 255$ km for three orders of derivative of $T$.

Good agreement achieved when SHS with converted coefficients and EHS were used.

When validation uses the global approximation of the ground data, EH and SH(converted) ”suit” more to this task.

Global gravity field models based on the ellipsoidal analysis might have principal advantages (e.g. EGM08).
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Thank you!
Synthesis of $|\nabla T^s|$, $|\nabla T^e|$, $u = b + 255 \text{ km}, N_{max} = 180$

\[
|\nabla T^s|^2 = \left( \frac{\partial T}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial T}{\partial \theta} \right)^2 + \left( \frac{1}{r \sin \theta} \frac{\partial T}{\partial \lambda} \right)^2
\]

\[
|\nabla T^e|^2 = \left( \frac{1}{w} \frac{\partial T}{\partial u} \right)^2 + \left( \frac{1}{w \sqrt{u^2 + E^2}} \frac{\partial T}{\partial \theta} \right)^2 + \left( \frac{1}{\sqrt{u^2 + E^2} \sin \vartheta} \frac{\partial T}{\partial \lambda} \right)^2
\]

\[
w = \sqrt{\frac{u^2 + E^2 \cos^2 \vartheta}{u^2 + E^2}}
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$|\nabla T^s|$ from SHS, $[m \cdot s^{-2}]$
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$|\nabla T_s| - |\nabla T_{\text{conv}}|$, RMS = 0.307 mGal
Synthesis of $|\nabla T^s|$, $|\nabla T^e|$, $u = b + 255$ km, $N_{max} = 180$

$|\nabla T_{conv}^s| - |\nabla T^e|$, RMS = 0.001 mGal
\[
T_{\bar{z}} = \frac{GM}{a} \frac{v}{L} \sum_{n,m} \frac{\partial S_{nm}(\frac{u}{E})}{S_{nm}(\frac{b}{E})} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda) P_{nm}(\cos \vartheta)
\]

\[
T_{\bar{z}\bar{z}} = \frac{GM}{a} \frac{v^2}{L^2} \sum_{n,m} \frac{\partial^2 S_{nm}(\frac{u}{E})}{S_{nm}(\frac{b}{E})} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda) P_{nm}(\cos \vartheta)
\]

\[
- \frac{GM}{a} \frac{uE^2 \sin^2 \vartheta}{L^4} \sum_{n,m} \frac{\partial S_{nm}(\frac{u}{E})}{S_{nm}(\frac{b}{E})} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda) P_{nm}(\cos \vartheta)
\]

\[
+ \frac{GM}{a} \frac{E^2 \sin \vartheta \cos \vartheta}{L^4} \sum_{n,m} \frac{S_{nm}(\frac{u}{E})}{S_{nm}(\frac{b}{E})} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda) \frac{\partial P_{nm}(\cos \vartheta)}{\partial \vartheta}
\]

\[
v^2 = u^2 + E^2
\]

\[
L^2 = u^2 + E^2 \cos^2 \vartheta
\]